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Tight Welfare Guarantees for Pure Nash Equilibria of the Uniform Price Auction

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Abstract
We revisit the inefficiency of the uniform price auction, one of the standard multi-unit auction formats, for allocating multiple units of a single good. In the uniform price auction, each bidder submits a sequence of non-increasing marginal bids, for each additional unit, i.e., a submodular curve. The per unit price is then set to be the highest losing bid. We focus on the pure Nash equilibria of such auctions, for bidders with submodular valuation functions. Our result is a tight upper and lower bound on the inefficiency of equilibria, showing that the Price of Anarchy is bounded by 2.1885. This resolves one of the open questions posed in previous works on multi-unit auctions. We also discuss implications of our bounds for an alternative, more practical form of the auction, employing a "*uniform bidding*" interface.

1 Introduction

Multi-unit auctions form a popular tool for selling multiple identical units of a single good. They have been in use for a long time, with important applications, such as

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the auctions offerred by the U.S. and U.K. Treasuries for selling bonds to investors. They are also being deployed in various platforms, including several online brokers [14, 15]. In the literature, multi-unit auctions have been a subject of study ever since the seminal work of Vickrey [20], and some formats were conceived even earlier, by Friedman [9].

We study the inefficiency of pure Nash equilibria of the uniform price auction, one of the standard multi-unit auction formats (see e.g. chapter 12 in [12]), for bidders with submodular valuation functions. In the canonical form of such an auction with k units, each bidder is required to submit a sequence of non-increasing bids, one for each additional unit, that is, a submodular curve. Among all submitted bids, the k highest win the auction and each bidder receives as many units of the good as the number of his winning bids. The highest losing bid is then chosen as the uniform price that each bidder pays per unit. A more practically relevant form of the auction employs a *uniform bidding interface*, which requires that each bidder submits a single *per-unit* bid, along with an upper bound on the number of units that this bid applies for. Note that this bidding interface is less expressive than the standard one described above, in that it essentially restricts the bidders' declarations to *"capped-linear"* curves, while their valuation functions can be submodular.

The simplicity of the uniform price auction is counterbalanced by the fact that it does not support truthful bidding in dominant strategies, thus encouraging strategic behavior. Even further, the underlying strategic game induced by the auction is prone to a *demand reduction* effect discussed in [1], where bidders may have incentives to understate their demand, so as to receive fewer units of the single good at a lower price per unit. This motivates the quantification of the inefficiency at equilibrium, via upper and lower bounds on the Price of Anarchy, which has been the subject of recent works [5, 7, 13, 19]. The outcomes of these works are quite encouraging, as they establish that the inefficiency is bounded by a small constant, under either full or incomplete information for the valuation functions. However, tight bounds for the Price of Anarchy are known only for a strict subset of pure Nash equilibria [13]. The results of [7] show that for pure Nash equilibria, the Price of Anarchy is bounded between $2 - \frac{1}{k}$ and 3.1462.

Contribution We focus on the pure Nash equilibria of the uniform price auction, for bidders with submodular valuation functions. Our results are tight upper and lower bounds on the inefficiency of pure equilibria, showing that the Price of Anarchy is bounded by 2.1885 for submodular bidders. This resolves one of the questions left open from [7, 13].

As was the case in these works and in much of the related work, our results concern pure Nash equilibrium profiles wherein no bidder outbids his actual value for any number of units (it is well known that without this condition, there exist equilibria that involve overbidding and have unbounded inefficiency, even for the single item second price auction). However, in contrast to the analysis presented in [7, 13], we do not restrict the bidders' strategy spaces to such "*no-overbidding*" strategies; that is, strategy profiles involving overbidding are perfectly valid for the game induced by the auction. Interestingly, we use such a potentially overbidding strategy explicitly in upper bounding the Price of Anarchy of no-overbidding pure equilibria. We argue formally in Section 2 that, for submodular bidders, all pure Nash equilibria of the game considered in [7, 13] survive in the game with unrestricted strategy spaces that we study here. The converse holds trivially as well, thus, the two versions of the game admit exactly the same set of no-overbidding equilibria. The proof of the upper bound on the Price of Anarchy in Section 3, is based on carefully analyzing the performance of (no-overbidding) pure equilibria with respect to bidders who receive fewer units than in the optimal assignment. Finally, in Section 3.1, we discuss the case of the simplified uniform bidding interface. An argument already discussed in [7] (extended version), makes our upper bound valid for no-overbidding equilibria of the auction under uniform bidding as well. However, we also provide a simple proof of an improved upper bound of 2, for uniform bidding with only 2 bidders.

Our lower bound in Section 4 applies for the standard bidding interface and is obtained by an explicit construction, involving 2 bidders with appropriately designed valuation functions, and a carefully chosen equilibrium profile. This equilibrium exhibits Price of Anarchy that approaches our upper bound of 2.1885, as the number of units on sale becomes large enough. Hence, this implies a clear separation of the performance of the two bidding interfaces for the case of 2 bidders, which is, perhaps surprisingly, in favor of the—less expressive—uniform bidding. This begs further study of upper bounds for the latter, towards determining whether this performance separation stands for more than 2 bidders.

1.1 Related Work

The uniform price auction is known to possess pure Nash equilibria, and a polynomial time algorithm for computing such an equilibrium is developed and analyzed in [13]. A tight bound of $e/(e-1)$ was obtained in [13], for the Price of Anarchy of pure equilibria *in undominated strategies*; these equilibria form a strict subset of all possible no-overbidding pure Nash equilibrium profiles. Indicatively, the auction always admits a welfare-optimal pure equilibrium, as described in [13], which lies outside this subset. For the full set of pure no-overbidding equilibria, the results of de Keijzer et al. in [7] imply that, with k units on sale, the Price of Anarchy is bounded between $2-\frac{1}{k}$ and 3.1462. Both of these bounds were shown in [7] to hold for the uniform bidding interface as well. The lower bound in particular is based on a pure equilibrium profile for the standard bidding interface, which consists of uniform bids (cf. Theorem 1 in the extended version of [7]). The upper bound was shown for the standard bidding interface, but is valid under uniform bidding as well, by virtue of a simple argument stating that, every pure equilibrium profile under uniform bidding is also an equilibrium profile under standard bidding. For completeness, we comment briefly on this argument in Section 3.1, as it renders our bound of 2.1885 valid also for the uniform bidding interface.

The social inefficiency of the uniform price auction has also been studied within the more general context of incomplete information. The upper bound of 3.1462 was shown in [7] to hold for mixed Bayes-Nash equilibria of the uniform price auction, in the incomplete information setting. This bound improved upon previously derived upper bounds of 4 from [13] and $4e/(e-1)$ from [19]. However, it is still not known whether this is tight for mixed Bayes-Nash equilibria. In [7], an upper bound of 4 is also obtained for the mixed Bayes-Nash Price of Anarchy, when the bidders have *subadditive* valuation functions, which form a strict superclass of submodular ones. This upper bound is derived by an appropriate adaptation of a technique introduced by Feldman et al. in [8] and is valid for the standard bidding interface. For the uniform bidding interface and bidders with subadditive valuation functions, an upper bound of 6.2924 is obtained, via approximating the values of subadditive valuation functions by uniform bids.

The uniform price auction constitutes an *item bidding* format, where each bidder casts a separate bid for each item (here: additional unit) he wishes to obtain. Another such multi-unit format is the discriminatory auction, studied in [5, 7]. The Price of Anarchy of item bidding formats has been studied extensively under full and incomplete information, particularly in the context of combinatorial auctions, implemented as simultaneous item bidding compositions of simple first or second price auctions. This line of research was initiated by Christodoulou, Kovács and Schapira in [4] and followed up by several other works, including e.g., [2, 5, 8, 10, 16, 19]. Roughgarden in [16, 17] and Syrgkanis and Tardos in [19] developed a *smoothness technique* that can be used for deriving upper bounds on the Price of Anarchy of simultaneous and sequential item bidding compositions of certain types of auctions. We refer the interested reader to [18], for a comprehensive survey of the relevant techniques and results. Overall, the outcomes of all these works highlight that several simple and practical mechanisms perform extremely well with respect to the social welfare attained at equilibrium.

2 Definitions and Preliminaries

We consider a multi-unit auction, involving the allocation of k units of a single item, to a set $\mathcal{N}$ of n bidders, $\mathcal{N} = \{1, \dots, n\}$. Each bidder $i \in \mathcal{N}$ has a private symmetric valuation function $v_i : \{0, 1, \dots, k\} \mapsto \mathbb{R}^+$, defined over the quantity of units that he receives, with $v_i(0) = 0$. In this work, we assume that each function v_i is a non-decreasing submodular function.

Definition 1 A valuation function $f : \{0, 1, \dots, k\} \mapsto \mathbb{R}^+$ is called **submodular** if for every $x < y$, $f(x) - f(x-1) \geqslant f(y) - f(y-1)$.

A valuation function can also be specified through a sequence of *marginal values*, corresponding to the value that each additional unit yields for the bidder. For the j-th additional unit, the bidder obtains marginal value $v_i(j) - v_i(j-1)$, which we denote by m_{ij}. Then, the function v_i can be determined by the vector $\mathbf{m}_i = (m_{i1}, \dots, m_{ik})$. For submodular functions, $m_{i1} \geqslant \cdots \geqslant m_{ik}$, by definition. We will often use the representation of v_i by $\mathbf{m}_i$ in the sequel.

The following proposition describes well known properties of submodular functions, and for the sake of completeness, we also provide their proof.

Proposition 1 *Given $x, y \in \{0, 1, \dots, k\}$ with $x \leqslant y$, any non-decreasing submodular function f, with $f(0) = 0$, satisfies $yf(x) \geqslant xf(y)$. Moreover, when*

$x < y$, for any $j = 1, \ldots, y - x$, the function f satisfies: $(f(x + j) - f(x))/j \geqslant (f(y) - f(x))/(y - x)$.

Proof Consider $x, y \in \{0, 1, \ldots, k\}$ with $y \geqslant x$. When $x = 0$, the first statement of the proposition holds, for a non-decreasing submodular function f with $f(0) = 0$, because $yf(x) = xf(y) = 0$. When $x \geqslant 1$, we can express $xf(y)$ as follows, by using the marginal values of f:

$$xf(y) = x\left(f(x) + \sum_{\ell=x+1}^{y} m_\ell\right) = xf(x) + x\sum_{\ell=x+1}^{y} m_\ell$$
$$\leqslant xf(x) + x(y - x)m_x \leqslant xf(x) + (y - x)x\frac{f(x)}{x} = yf(x)$$

The first inequality above is due to the non-increasing marginal values, i.e., that $m_x \geqslant m_\ell$, for $\ell = x + 1, \ldots, y$. The second inequality is justified by the fact that $m_x \leqslant m_\ell$ for all $\ell = 1, \ldots, x$, thus, $m_x \leqslant f(x)/x$, which is the average of these marginal values.

For the second statement of the proposition, consider $y > x$ and $j = 1, \ldots, y - x$. Define the function $g(j) = f(x + j) - f(x)$, over $\{1, \ldots, y - x\}$. Using the fact that f is submodular non-decreasing, it can be straightforwardly verified that g is submodular non-decreasing as well, by Definition 1. Then it satisfies the first statement of the proposition, i.e., $(y-x)g(j) \geqslant jg(y-x)$, for any $j = 1, \ldots, y-x$, which is precisely the statement we wanted to prove. □

The standard uniform price auction requires that bidders submit their non-increasing marginal value for each additional unit; every bidder i is required to declare his valuation curve, as a *bid vector* $\mathbf{b}_i = (b_{i1}, b_{i2}, \ldots, b_{ik})$, satisfying $b_{i1} \geqslant b_{i2} \geqslant \cdots \geqslant b_{ik}$. Thus, b_{ij} is the *declared* marginal value of i, for obtaining the j-th unit of the item, if he has already obtained $j - 1$ units. We sometimes refer to the bids b_{ij} as the marginal bids of i. Note that each b_{ij} may differ from the bidder's actual marginal value, m_{ij}. Given a *bidding profile* $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, the auction allocates the k units to the k highest marginal bids. We denote this allocation by $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), x_2(\mathbf{b}), \ldots, x_n(\mathbf{b}))$, where $x_i(\mathbf{b})$ is the number of units allocated to bidder i. Each bidder pays a uniform price $p(\mathbf{b})$ *per received unit*, which equals the highest rejected marginal bid, i.e., the $(k + 1)$-th highest marginal bid. The total payment of bidder i then equals $x_i(\mathbf{b}) \cdot p(\mathbf{b})$, and his utility for the allocation is: $u_i(\mathbf{b}) = v_i(x_i(\mathbf{b})) - x_i(\mathbf{b}) \cdot p(\mathbf{b})$.

The (utilitarian) *social welfare* achieved by the auction under a bidding profile $\mathbf{b}$ is defined as the sum of utilities of all interacting parties, inclusively of the auctioneer's revenue. This sum equals the sum of the bidders' values for their allocations:

$$SW(\mathbf{b}) = \sum_{i=1}^{n} v_i(x_i(\mathbf{b}))$$

We will also denote by $SW(\mathbf{x})$ the social welfare produced by an allocation $\mathbf{x} = (x_1, \ldots, x_n)$, of the units to the players. Our goal is to derive upper and lower bounds on the *Price of Anarchy* (PoA) [11] of pure Nash equilibria of the uniform price auction. This is the worst-case ratio of the optimal welfare, over the welfare achieved at a pure Nash equilibrium. If $\mathbf{x}^*$ denotes an optimal allocation, then

$$PoA = \sup_{\mathbf{b}} \frac{SW(\mathbf{x}^*)}{SW(\mathbf{b})}$$

where the supremum is taken over pure equilibrium profiles.

Following previous works on equilibrium analysis of auctions, e.g., [2, 4, 13], we focus on *no-overbidding* equilibrium profiles $\mathbf{b}$, wherein no bidder ever outbids his value, for any number of units. That is, if $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ is an equilibrium, then for any bidder i, and for any $\ell \leqslant k$, we assume $\sum_{j=1}^{\ell} b_{ij} \leqslant v_i(\ell)$. Note that, this *does not* necessarily imply $b_{ij} \leqslant m_{ij}$, except for when $j = 1$: i.e., $b_{i1} \leqslant m_{i1} = v_i(1)$. We also stress that, in contrast to previous works, we do not restrict the bidders' strategy spaces to no-overbidding strategies. Still, for submodular valuation functions, a pure Nash equilibrium profile $\mathbf{b}$ of the game restricted to no-overbidding strategies is also a pure Nash equilibrium for the game with unrestricted strategies. Indeed, towards deriving a contradiction, assume that for an equilibrium $\mathbf{b}$ of the restricted game, there is a bidder i and an overbidding deviation strategy $\mathbf{b}'_i$ such that $u_i(\mathbf{b}') > u_i(\mathbf{b}) \geqslant 0$, where $\mathbf{b}' \equiv (\mathbf{b}'_i, \mathbf{b}_{-i})$. Then, $v_i(x_i(\mathbf{b}')) > x_i(\mathbf{b}')p(\mathbf{b}')$, which implies the existence of an equivalent deviation to $\mathbf{b}'_i$, having the first $x_i(\mathbf{b}')$ marginal bids equal to $v_i(x_i(\mathbf{b}'))/x_i(\mathbf{b}') > p(\mathbf{b}')$ and the remaining ones equal to 0 (note that $x_i(\mathbf{b}') > 0$ since we assumed that $\mathbf{b}'_i$ is a profitable deviation); this strategy also wins exactly $x_i(\mathbf{b}')$ units when the other bidders stick to $\mathbf{b}_{-i}$, since all the non-zero bids of bidder i are higher than the previous price $p(\mathbf{b}')$. By the submodularity of v_i, this new strategy satisfies no-overbidding[1]—a contradiction to $\mathbf{b}$ being an equilibrium for the no-overbidding game.

In proving our upper bound on the Price of Anarchy of no-overbidding pure Nash equilibrium profiles $\mathbf{b}$, we assume that the bids of every bidder i in any such equilibrium profile $\mathbf{b}$ sum up to his value, $v_i(x_i(\mathbf{b}))$. This assumption does not harm generality, as shown below.[2]

Proposition 2 *Let* $\mathbf{b}$ *denote a no-overbidding pure Nash equilibrium profile of the uniform price auction. There exists a (no-overbidding) pure Nash equilibrium profile* $\mathbf{b}'$ *such that, for every bidder* i*,* $x_i(\mathbf{b}') = x_i(\mathbf{b})$ *and:*

1. $b'_{ij} = 0$*, for* $j \geqslant x_i(\mathbf{b}') + 1$,
2. $\sum_{j=1}^{k} b'_{ij} = v_i(x_i(\mathbf{b}'))$.

Proof First we show how to transform $\mathbf{b}$ into a no-overbidding equilibrium profile $\mathbf{b}^0$ satisfying, for every bidder i: $x_i(\mathbf{b}^0) = x_i(\mathbf{b})$ and $b^0_{ij} = 0$, for $j \geqslant x_i(\mathbf{b}^0) + 1$. For

[1] By Proposition 1, for every $\ell \leqslant x_i(\mathbf{b}')$: $\ell \cdot (v_i(x_i(\mathbf{b}'))/x_i(\mathbf{b}')) \leqslant v_i(\ell)$.

[2] The fact that this assumption can be made without loss of generality is instrumental in the proof of Lemma 3, which lies at the heart of our proof for the upper bound. Although we implicitly made this same assumption in our preliminary conference proceedings version of this work [3], a formal statement of this was omitted.

any bidder i under $\mathbf{b}$, each of his "*non-winning*" marginal bids b_{ij}, $j \geqslant x_i(\mathbf{b}) + 1$, cannot exceed the uniform price under $\mathbf{b}$, i.e., $b_{ij} \leqslant p(\mathbf{b})$. We produce $\mathbf{b}^0$ by lowering all these marginal bids down to 0 for all the bidders. This modification reduces the uniform price to $p(\mathbf{b}^0) = 0$ and increases the winning bidders' utilities. Then, clearly, no winning bidder has an incentive to "*drop*" any of his won units. Moreover, if *any* bidder i deviates—say, through a strategy $\bar{\mathbf{b}}_i$—towards obtaining *more* than $x_i(\mathbf{b}^0) = x_i(\mathbf{b})$ units, the uniform price will necessarily increase, to at least the value of the smallest winning marginal bid under $\mathbf{b}^0$; this equals the smallest winning marginal bid under $\mathbf{b}$. The deviating bidder's utility will then become $u_i(\bar{\mathbf{b}}_i, \mathbf{b}^0_{-i}) = u_i(\bar{\mathbf{b}}_i, \mathbf{b}_{-i}) \leqslant u_i(\mathbf{b}) \leqslant u_i(\mathbf{b}^0)$. Thus, $\mathbf{b}^0$ is a pure Nash equilibrium and satisfies the first of the stated properties.

Let us now describe a procedure that transforms an equilibrium profile $\mathbf{b}$ satisfying the first property, into a profile $\mathbf{b}'$ satisfying the second stated property as well. Begin by setting $\mathbf{b}' = \mathbf{b}$. Subsequently, for every winning bidder i with $x_i(\mathbf{b}') \geqslant 1$, adjust his marginal bids b_{ij} iteratively, for $j = 1, \dots, x_i(\mathbf{b}')$, by increasing b'_{ij} to:

$$b'_{ij} + \max\left\{0,\ \min\left\{m_{ij} - b'_{ij},\ v_i(x_i(\mathbf{b}')) - \sum_{j=1}^{x_i(\mathbf{b})} b'_{ij}\right\}\right\}$$

Consider for example a bidder i with valuation function $v_i = (2, 1, 0, \dots, 0)$, bidding $\mathbf{b}_i = (1.5, 1.5, 0, \dots, 0)$ in some equilibrium profile $\mathbf{b}$, and obtaining $x_i(\mathbf{b}) = 2$ units. The procedure outlined above initializes $\mathbf{b}'_i = \mathbf{b}_i$; in the first iteration, it computes $\min\{m_{i1} - b'_{i1},\ v_i(2) - b'_{i1} - b'_{i2}\} = \min\{0.5, 0\} = 0$. Thus, b'_{i1} remains unchanged and equal to 1.5. In the second iteration, we have $\min\{m_{i2} - b'_{i2},\ v_i(2) - b'_{i1} - b'_{i2}\} = \min\{-0.5, 0\} = -0.5$. Thus, b'_{i2} also remains unchanged and equal to 1.5. For the same valuation function and bidding vector $\mathbf{b}_i = (1.6, 0.8, 0, \dots, 0)$ it can be verified that the procedure outputs $\mathbf{b}'_i = (2, 1, 0, \dots, 0)$; similarly, for a bidding vector $\mathbf{b}_i = (1.4, 1.1, 0, \dots, 0)$ it outputs $\mathbf{b}'_i = (1.9, 1.1, 0, \dots, 0)$. Notice that all the input and output vectors satisfy no-overbidding.

This transformation increases each of the winning marginal bids of each winning bidder as much as possible, but not higher than the bidder's corresponding marginal value, by "*consuming*" iteratively the remaining difference between his value for the number of the received units, $x_i(\mathbf{b}') = x_i(\mathbf{b})$, and the current sum of his bids. The uniform price remains unchanged, i.e., $p(\mathbf{b}') = p(\mathbf{b}) = 0$, implying $u_i(\mathbf{b}') = u_i(\mathbf{b})$, for every bidder i. Thus, no winning bidder has an incentive to "*drop*" any units won under $\mathbf{b}'$. Moreover, no other bidder has an incentive to deviate—say, through a strategy $\bar{\mathbf{b}}_i$—towards obtaining more units, as the prospective uniform price $p(\bar{\mathbf{b}}_i, \mathbf{b}'_{-i})$ will equal a winning marginal bid under $\mathbf{b}'$, thus, will be at least equal to $p(\bar{\mathbf{b}}_i, \mathbf{b}_{-i})$ implying $u_i(\bar{\mathbf{b}}_i, \mathbf{b}'_{-i}) \leqslant u_i(\bar{\mathbf{b}}_i, \mathbf{b}_{-i}) \leqslant u_i(\mathbf{b}) = u_i(\mathbf{b}')$. □

3 Inefficiency Upper Bound

In this section we develop tight welfare guarantees for no-overbidding pure Nash equilibrium profiles of the uniform price auction, when the bidders have submodular valuation functions. We also provide upper bounds for the simplified uniform bidding interface of this auction. By the results of [7], it is already known that for submodular valuation functions on k units, $2 - \frac{1}{k} \leqslant PoA \leqslant 3.146$. We show that:

Theorem 1 *The Price of Anarchy of no-overbidding pure Nash equilibria of the uniform price auction with submodular bidders is at most:*

$$\frac{2 + \mathcal{W}_0(-e^{-2})}{1 + \mathcal{W}_0(-e^{-2})} \approx 2.1885$$

where $\mathcal{W}_0$ is the first branch of the Lambert W function.

The Lambert W function is the multi-valued inverse function of $f(W) = We^W$; we refer the reader to [6] for an informative exposition of its definition, properties and applications.

Let $\mathbf{b}$ denote a no-overbidding profile. For the remainder of the analysis in this section, we focus without loss of generality on pure equilibria that result after applying Proposition 2. We denote the winning (marginal) bids under $\mathbf{b}$ by $\beta_j(\mathbf{b})$, $j = 1, \dots, k$, so that $\beta_j(\mathbf{b})$ is the j-th lowest winning bid under $\mathbf{b}$, thus, $\beta_1(\mathbf{b}) \leqslant \beta_2(\mathbf{b}) \leqslant \cdots \leqslant \beta_k(\mathbf{b})$. For a profile of valuation functions $(v_1, v_2, \dots, v_n)$, we denote the *socially optimal*—i.e., welfare maximizing—allocation by $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$. If there are multiple such allocations, we fix one for the remainder of the analysis. Given $\mathbf{x}^*$ and any other arbitrary allocation $\mathbf{x}$, we define a partition of the set of bidders, $\mathcal{N}$, into two subsets, $\mathcal{O}$ and $\mathcal{U}$, as follows:

$$\mathcal{N} = \mathcal{O} \cup \mathcal{U}, \quad \mathcal{O} = \{i \in \mathcal{N} : x_i \geqslant x_i^*\}, \quad \mathcal{U} = \{i \in \mathcal{N} : x_i < x_i^*\}.$$

The set $\mathcal{O}$ contains the "*overwinners*", i.e., bidders who receive in $\mathbf{x}$ at least as many units as in $\mathbf{x}^*$. The set $\mathcal{U}$ contains respectively the "*underwinners*". In our analysis, the allocations we refer to are determined by some profile $\mathbf{b}$, i.e., $\mathbf{x} \equiv \mathbf{x}(\mathbf{b})$. Consequently, the sets $\mathcal{O}$ and $\mathcal{U}$ will depend on $\mathbf{b}$; for simplicity, we omit this dependence from our notation. The following lemma states that, under a no-overbidding profile $\mathbf{b}$, every bidder $i \in \mathcal{O}$ retains value at least equal to a convex combination of her socially optimal value, $v_i(x_i^*)$, and of the sum of her winning bids.

Lemma 1 *Let* $\mathbf{b}$ *be a no-overbidding bidding profile, and let $\mathcal{O}$ be the set of overwinners with respect to the allocation* $\mathbf{x}(\mathbf{b})$*. Then, for every $\lambda \in [0, 1]$, and for every bidder $i \in \mathcal{O}$:*

$$v_i(x_i(\mathbf{b})) \geqslant \lambda \cdot v_i(x_i^*) + (1 - \lambda) \cdot \sum_{j=1}^{x_i(\mathbf{b})} b_{ij} \tag{1}$$

Proof Indeed, by definition of $\mathcal{O}$:

$$v_i(x_i(\mathbf{b})) = \lambda v_i(x_i(\mathbf{b})) + (1-\lambda)v_i(x_i(\mathbf{b})) \geqslant \lambda v_i(x_i^*) + (1-\lambda)v_i(x_i(\mathbf{b}))$$

Then, (1) follows by our no-overbidding assumption on $\mathbf{b}$. □

By definition, each overwinner is capable of "*covering*" her socially optimal value. Conversely, the underwinners are the cause of social inefficiency. We will bound the total inefficiency by transforming the leftover fractions of winning bids of bidders in $\mathcal{O}$, i.e., the term $(1-\lambda)\cdot\sum_{j=1}^{x_i(\mathbf{b})} b_{ij}$ for each bidder $i \in \mathcal{O}$ in (1), into fractions of the value attained by bidders in $\mathcal{U}$ in the optimal allocation. In this manner, we will quantify the value that the underwinners are missing (due to their strategic bidding), and determine the worst-case scenario that can arise at a pure Nash equilibrium. The following claim can be inferred from [13], and will be used to facilitate this transformation. We present the proof for completeness.

Claim 1 *Let* $\mathbf{b}$ *be any bidding profile. Then it holds that:*

$$\sum_{i\in\mathcal{U}} \sum_{j=1}^{x_i^*-x_i(\mathbf{b})} \beta_j(\mathbf{b}) \leqslant \sum_{i\in\mathcal{O}} \sum_{j=x_i^*+1}^{x_i(\mathbf{b})} b_{ij}. \tag{2}$$

Proof For every unit missed under $\mathbf{b}$ by any bidder $i \in \mathcal{U}$ (with respect to the units won by i in the optimal allocation), there must exist some bidder $\ell \in \mathcal{O}$ that obtains this unit. If i missed $x_i^* - x_i(\mathbf{b}) > 0$ units under $\mathbf{b}$, there are at least as many bids issued by bidders in $\mathcal{O}$ who obtained collectively these units. The sum of these bids cannot be less than the sum $\sum_{j=1}^{x_i^*-x_i(\mathbf{b})} \beta_j(\mathbf{b})$ of the $x_i^* - x_i(\mathbf{b})$ lowest winning bids in $\mathbf{b}$. Hence, summing over every $i \in \mathcal{U}$ yields the desired inequality. □

Next, we develop a characterization of upper bounds on the Price of Anarchy. To this end, let us first define the following set, $\Lambda(\mathbf{b})$, for any bidding profile $\mathbf{b}$.

$$\Lambda(\mathbf{b}) = \left\{ \lambda \in [0,1] : \; v_i(x_i(\mathbf{b})) + (1-\lambda) \sum_{j=1}^{x_i^*-x_i(\mathbf{b})} \beta_j(\mathbf{b}) \geqslant \lambda v_i(x_i^*), \; \forall i \in \mathcal{U} \right\} \tag{3}$$

Notice that, for every $\mathbf{b}$, $\Lambda(\mathbf{b}) \neq \emptyset$, because $\lambda = 0 \in \Lambda(\mathbf{b})$. The following simple lemma helps us understand how one can obtain upper bounds on the Price of Anarchy.

Lemma 2 *If there exists* $\lambda \in [0,1]$ *such that* $\lambda \in \Lambda(\mathbf{b})$, *for every no-overbidding pure Nash equilibrium profile* $\mathbf{b}$ *of the uniform price auction, then the Price of Anarchy of no-overbidding pure Nash equilibria is at most* λ^{-1}.

Proof Fix a no-overbidding pure Nash equilibrium profile $\mathbf{b}$ and consider any $\lambda \in \Lambda(\mathbf{b})$. Then, we can apply consecutively the partition $\mathcal{N} = \mathcal{O} \cup \mathcal{U}$ with respect to $\mathbf{b}$, Lemma 1, Claim 1, and finally, the definition of $\Lambda(\mathbf{b})$, to obtain:

$$\begin{aligned} SW(\mathbf{b}) &= \sum_{i\in\mathcal{O}} v_i(x_i(\mathbf{b})) + \sum_{i\in\mathcal{U}} v_i(x_i(\mathbf{b})) \\ &\geqslant \lambda \sum_{i\in\mathcal{O}} v_i(x_i^*) + (1-\lambda)\sum_{i\in\mathcal{O}} \sum_{j=x_i^*+1}^{x_i(\mathbf{b})} b_{ij} + \sum_{i\in\mathcal{U}} v_i(x_i(\mathbf{b})) \\ &\geqslant \lambda \sum_{i\in\mathcal{O}} v_i(x_i^*) + \sum_{i\in\mathcal{U}} \left((1-\lambda) \sum_{j=1}^{x_i^*-x_i(\mathbf{b})} \beta_j(\mathbf{b}) + v_i(x_i(\mathbf{b})) \right) \\ &\geqslant \lambda \sum_{i\in\mathcal{O}} v_i(x_i^*) + \sum_{i\in\mathcal{U}} \lambda \cdot v_i(x_i^*) = \lambda \cdot SW(\mathbf{x}^*) \end{aligned}$$

□

Using $\lambda = 0$ with Lemma 2, yields the trivial upper bound of ∞. To obtain better upper bounds, Lemma 2 shows that we need to understand better the sets $\Lambda(\mathbf{b})$, and whether underwinners can extract at equilibrium a good fraction of their value under the optimal assignment. By the definition of these sets, the next step towards this is to derive lower bounds on every $\beta_\ell(\mathbf{b})$ for each underwinner $i \in \mathcal{U}$, and every value $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$. The lower bound that we will use is formally expressed below.

Lemma 3 *Let* $\mathbf{b}$ *be a pure Nash equilibrium of the uniform price auction, and let* $\mathbf{x}^*$ *be a socially optimal allocation. For every underwinning bidder* $i \in \mathcal{U}$ *under* $\mathbf{b}$ *and for every* $\ell = 1, \cdots, x_i^* - x_i(\mathbf{b})$*:*

$$\beta_\ell(\mathbf{b}) \geqslant \frac{1}{x_i(\mathbf{b}) + \ell} \cdot \left(v_i(x_i(\mathbf{b}) + \ell) - v_i(x_i(\mathbf{b}))\right) \tag{4}$$

For completeness, we note that (4) was also derived and used in [13] (cf. proof of Theorem 2), for a narrower class of pure Nash equilibria and under the restriction of no-overbidding on the bidders' strategy spaces; its proof here differs significantly. We defer the proof of Lemma 3, in order to explain first how it—along with Lemma 2—leads to the proof of Theorem 1.

Proof of Theorem 1 In order to apply Lemma 2, we identify values of λ that belong to every set $\Lambda(\mathbf{b})$, induced by a no-overbidding pure Nash equilibrium profile $\mathbf{b}$. Fix any such no-overbidding pure Nash equilibrium profile $\mathbf{b}$ and, for every bidder $i \in \mathcal{U}$, let $q_i(\mathbf{b}) = x_i^* - x_i(\mathbf{b})$. To simplify the notation, we use hereafter x_i for $x_i(\mathbf{b})$, p for $p(\mathbf{b})$, q_i for $q_i(\mathbf{b})$, and β_j for $\beta_j(\mathbf{b})$, (always with respect to the no-overbidding pure Nash equilibrium $\mathbf{b}$).

For every $\lambda \in [0, 1]$ and every $i \in \mathcal{U}$, define $h_i(\lambda) = v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \beta_j$. We can now have the following implications.

$$\begin{aligned} h_i(\lambda) &= v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \beta_j \\ &\geqslant v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \frac{1}{j + x_i} \cdot (v_i(x_i + j) - v_i(x_i)) \qquad (5) \\ &= v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \left(\frac{j}{j + x_i} \cdot \frac{v_i(x_i + j) - v_i(x_i)}{j} \right) \\ &\geqslant v_i(x_i) + (1-\lambda) \cdot \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \cdot \sum_{j=1}^{q_i} \frac{j}{j + x_i}. \qquad (6) \end{aligned}$$

In the derivation above, inequality (5) follows by applying (4) from Lemma 3, for every β_j, $j = 1, \ldots, q_i$. Inequality (6) follows by application of the second statement of Proposition 1, which yields $\frac{v_i(x_i+j)-v_i(x_i)}{j} \geqslant \frac{v_i(x_i^*)-v_i(x_i)}{x_i^*-x_i}$, for any $j = 1, \ldots, q_i$.

Suppose now that under the equilibrium $\mathbf{b}$, there exists $i \in \mathcal{U}$ such that $x_i = 0$. In order for some λ to belong to $\Lambda(\mathbf{b})$, we would need to have $h_i(\lambda) \geqslant \lambda v_i(x_i^*)$. Using (6), for the underwinners with $x_i = 0$, and substituting $v_i(x_i) = 0$, we obtain: $h_i(\lambda) \geqslant (1-\lambda) v_i(x_i^*)$. For any $\lambda \leqslant 1/2$, it is true that $(1-\lambda) v_i(x_i^*) \geqslant \lambda v_i(x_i^*)$. Thus, any value of λ in $[0, 1/2]$ satisfies the constraint in the definition of $\Lambda(\mathbf{b})$ for bidders in $\mathcal{U}$ with $x_i = 0$. It remains to consider the more interesting case, which is of bidders in $\mathcal{U}$ with $x_i > 0$. We continue from (6) to bound $h_i(\lambda)$ for those bidders as follows:

$$\begin{aligned} h_i(\lambda) &\geqslant \lambda v_i(x_i) + (1-\lambda) \cdot \left(v_i(x_i) + \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \cdot \sum_{j=1}^{q_i} \frac{j}{j + x_i} \right) \\ &\geqslant \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \frac{x_i}{x_i^* - x_i} \cdot \left(1 - \sum_{j=1}^{q_i} \frac{1}{j + x_i} \right) \right) \\ &\geqslant \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \frac{x_i}{x_i^* - x_i} \cdot \left(1 - \int_{x_i}^{x_i^*} \frac{1}{z} dz \right) \right) \\ &\geqslant \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \frac{x_i}{x_i^* - x_i} \cdot \left(1 + \ln \frac{x_i}{x_i^*} \right) \right) \\ &= \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \frac{\frac{x_i}{x_i^*}}{1 - \frac{x_i}{x_i^*}} \cdot \left(1 + \ln \frac{x_i}{x_i^*} \right) \right) \qquad (7) \end{aligned}$$

The second inequality follows from the fact that $v_i(x_i) \geqslant \frac{x_i}{x_i^*-x_i} \cdot \sum_{j=x_i+1}^{x_i^*} m_{ij}$, which is an implication of the first statement of Proposition 1. Note also that $\sum_{j=x_i+1}^{x_i^*} m_{ij} = v_i(x_i^*) - v_i(x_i)$. We have bounded the sum of harmonic terms by using $\sum_{k=m}^{n} f(k) \leqslant \int_{m-1}^{n} f(x)dx$, which holds for any monotonically decreasing positive function.

Having (7), we minimize the function $f(y) = 1 + \frac{y}{1-y} \cdot (1 + \ln y)$ over $(0, 1)$, since x_i/x_i^* belongs to this interval. Note that f is continuous and differentiable in $(0, 1)$, with derivative $f'(y) = \frac{-y+\ln y+2}{(1-y)^2}$. In the interval $(0, 1)$, we have $f'(y) = 0$ when $-y + \ln y + 2 = 0$ or, equivalently, when $-ye^{-y} = -e^{-2}$. This yields $y = -\mathcal{W}_0(-e^{-2})$, where $\mathcal{W}_0$ is the first branch of the Lambert W function. It can be verified that f is also convex in $(0, 1)$, by examination of the sign of f'' around $-\mathcal{W}_0(-e^{-2})$; to conclude, f is minimized at $y_0 = -\mathcal{W}_0(-e^{-2})$ and then, $1+\ln y_0 = y_0 - 1$, which yields $f(y_0) = 1 - y_0$.

By substituting in (7), we obtain a new lower bound on $h_i(\lambda)$ for every bidder $i \in \mathcal{U}$ with $x_i > 0$ as follows:

$$h_i(\lambda) \geqslant \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij}\right) \cdot \left(1 + \mathcal{W}_0(-e^{-2})\right)$$

We can now obtain candidate values of λ that belong to $\Lambda(\mathbf{b})$, if we set the right hand side of the above to be greater than or equal to $\lambda v_i(x_i^*)$. In particular, we notice that by using $\lambda^* = (1 + \mathcal{W}_0(-e^{-2}))/(2 + \mathcal{W}_0(-e^{-2})) \approx 0.457$, we have that $h_i(\lambda^*) \geqslant \lambda^* v_i(x_i^*)$[3] for every bidder $i \in \mathcal{U}$ with $x_i > 0$. Since for bidders with $x_i = 0$, we found earlier that $\lambda \leqslant 1/2$ suffices, and since $\lambda^* < 1/2$, we conclude that $\lambda^* \in \Lambda(\mathbf{b})$. Hence, the theorem follows by Lemma 2. ☐

To complete our analysis, we provide the proof of Lemma 3.

Proof of Lemma 3 Let $\mathbf{b}$ denote a no-overbidding pure Nash equilibrium profile $\mathbf{b}$ and $p(\mathbf{b})$ be the uniform price under $\mathbf{b}$. We assume that $\mathbf{b}$ satisfies the two properties outlined in Proposition 2. Fix now any underwinning bidder $i \in \mathcal{U}$. We examine a particular unilateral deviation of i from $\mathbf{b}$, towards obtaining ℓ additional units, for $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$. Given ℓ, define $r = \max\{ j \leqslant x_i(\mathbf{b}) : m_{ij} > \beta_\ell(\mathbf{b}) \}$ and consider the following deviation $\mathbf{b}_i'$ for i:

$$\mathbf{b}_i' = (\underbrace{m_{i1}, \cdots, m_{ir}}_{r \text{ bids}}, \underbrace{\beta_\ell(\mathbf{b}) + \epsilon, \beta_\ell(\mathbf{b}) + \epsilon, \ldots, \beta_\ell(\mathbf{b}) + \epsilon}_{x_i(\mathbf{b})+\ell-r \text{ bids}}, 0, 0, \ldots, 0)$$

The first part of $\mathbf{b}_i'$ consists of all $r \leqslant x_i(\mathbf{b})$ highest marginal values of i, that are higher than $\beta_\ell(\mathbf{b})$, i.e., the ℓ-th lowest winning bid in $\mathbf{b}$. Thus, either $r = x_i(\mathbf{b})$, or $r < x_i(\mathbf{b})$ and $m_{i,r+1} \leqslant \beta_\ell(\mathbf{b})$. These r marginal values are followed by $x_i(\mathbf{b}) - r + \ell$ bids equal to $\beta_\ell(\mathbf{b}) + \epsilon$, where $\epsilon > 0$ is *any* arbitrarily small constant, no larger than $m_{ir} - \beta_\ell(\mathbf{b})$. We examine the cases $r = x_i(\mathbf{b})$ and $r < x_i(\mathbf{b})$ separately.

[3]The following holds: $\lambda^* \cdot v_i(x_i(\mathbf{b})) + (1-\lambda^*) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij}\right) \cdot \left(1 + \mathcal{W}_0(-e^{-2})\right) = \frac{1+\mathcal{W}_0(-e^{-2})}{2+\mathcal{W}_0(-e^{-2})} \cdot v_i(x_i^*)$.

Case 1: $r = x_i(\mathbf{b})$. First, we claim that the bidding vector $\mathbf{b}'_i$ grants bidder i exactly $x_i(\mathbf{b}'_i, \mathbf{b}_{-i}) = x_i(\mathbf{b}) + \ell$ units in total, under the profile $(\mathbf{b}'_i, \mathbf{b}_{-i})$. If this is not the case, it is implied that at least one of the marginal bids $\beta_1(\mathbf{b}), \ldots, \beta_\ell(\mathbf{b})$ under $\mathbf{b}$ belongs to bidder i (otherwise, all the $x_i(\mathbf{b}) + \ell$ non-zero marginal bids of bidder i in $\mathbf{b}'_i$ would be winning bids). Then, however, since $r = x_i(\mathbf{b})$ and $m_{ir} > \beta_\ell(\mathbf{b}) \geqslant \cdots \geqslant \beta_1(\mathbf{b})$, and $\mathbf{b}$ is a no-overbidding equilibrium, we conclude that $v_i(x_i(\mathbf{b})) > \sum_{j=1}^{x_i(\mathbf{b})} b_{ij}$. This contradicts our assumption that $\mathbf{b}$ satisfies property 2 of Proposition 2.

Moreover, if $r = x_i(\mathbf{b})$, the uniform price under the profile $(\mathbf{b}'_i, \mathbf{b}_{-i})$ equals exactly $\beta_\ell(\mathbf{b})$, which becomes the highest losing bid (issued by some other bidder in the auction). Note that $\mathbf{b}'_i$ may constitute overbidding for i, but it is a legitimate action in our setting. Since $\mathbf{b}$ is a pure Nash equilibrium, the utility of the bidder at $(\mathbf{b}'_i, \mathbf{b}_{-i})$ cannot be higher than the utility obtained by the bidder at $\mathbf{b}$, i.e.:

$$v_i(x_i(\mathbf{b}) + \ell) - (x_i(\mathbf{b}) + \ell) \cdot \beta_\ell(\mathbf{b}) \leqslant v_i(x_i(\mathbf{b})) - x_i(\mathbf{b}) \cdot p(\mathbf{b})$$

By dropping the non-negative term $x_i(\mathbf{b}) \cdot p(\mathbf{b})$ and solving for $\beta_\ell(\mathbf{b})$, we obtain (4).

Case 2: $r < x_i(\mathbf{b})$. In this case the bidding vector $\mathbf{b}'_i$ definitely constitutes overbidding for i, since $m_{i,r+1} \leqslant \beta_\ell(\mathbf{b}) < \beta_\ell(\mathbf{b}) + \epsilon$; by the submodularity of the valuation function, $m_{ij} < \beta_\ell(\mathbf{b}) + \epsilon$ for $j \geqslant r + 1$, thus:

$$\begin{aligned}
v_i(x_i(\mathbf{b}) + \ell) &= \sum_{j=1}^{r} m_{ij} + \sum_{j=r+1}^{x_i(\mathbf{b})+\ell} m_{ij} \\
&< \sum_{j=1}^{r} m_{ij} + (x_i(\mathbf{b}) + \ell - r) \cdot (\beta_\ell(\mathbf{b}) + \epsilon) \\
&\leqslant \sum_{j=1}^{x_i(\mathbf{b})} m_{ij} + (x_i(\mathbf{b}) + \ell) \cdot (\beta_\ell(\mathbf{b}) + \epsilon) \\
&= v_i(x_i(\mathbf{b})) + (x_i(\mathbf{b}) + \ell) \cdot (\beta_\ell(\mathbf{b}) + \epsilon)
\end{aligned}$$

By rearranging, we obtain:

$$\beta_\ell(\mathbf{b}) > \frac{1}{\ell + x_i(\mathbf{b})} \cdot (v_i(x_i(\mathbf{b}) + \ell) - v_i(x_i(\mathbf{b}))) - \epsilon.$$

Observe that the above inequality holds for any arbitrarily small constant $\epsilon > 0$. Thus, inequality (4) follows and the proof is concluded. □

3.1 Uniform Bidding

We comment here on the uniform price auction with a uniform bidding interface. In this form of the auction, each bidder i submits a single per-unit bid b_i, along with an upper bound q_i on the number of units that his bid applies for. This can be simulated by a bidding vector $\mathbf{b}_i$, composed of q_i initial marginal bids equal to b_i, that are followed by $k - q_i$ zeros.

Lemma 1 of [7] (extended version) states that every pure Nash equilibrium profile of the auction under the uniform bidding interface remains a pure Nash equilibrium under the standard bidding interface. The argument is similar to the one we used in Section 2 for claiming that every (no-overbidding) pure Nash equilibrium $\mathbf{b}$ of the auction with no-overbidding strategies remains equilibrium for the auction with overbidding allowed (before the statement of Proposition 2). It suffices to assume additionally that the profile $\mathbf{b}$ consists of uniform bids and that, for some bidder i, there exists a deviation strategy $\mathbf{b}'_i$—which may involve overbidding, but is not restricted to consist of uniform bids—such that $u_i(\mathbf{b}'_i, \mathbf{b}_{-i}) > u_i(\mathbf{b})$. As we argued in Section 2, we end up with a no-overbidding *uniform bidding* strategy that is utility-wise equivalent to $\mathbf{b}'_i$ for i, a contradiction to $\mathbf{b}$ being an equilibrium with uniform bids for the game with no-overbidding strategies. This, along with our Theorem 1, implies:

Corollary 1 *The Price of Anarchy of no-overbidding pure Nash equilibria of the uniform price auction for submodular bidders under the uniform bidding interface is at most* $(2 + \mathcal{W}_0(e^{-2}))/(1 + \mathcal{W}_0(e^{-2})) \approx 2.1885$.

Interestingly, for 2 bidders only, we can improve on the upper bound of Corollary 1:

Proposition 3 *The Price of Anarchy of pure Nash equilibria of the uniform price auction under the uniform bidding interface is at most 2, for 2 submodular bidders.*

Proof Under a pure Nash equilibrium profile $\mathbf{b}$, let bidder i be the *underwinner* and bidder $j \neq i$ be the *overwinner*. To simplify the notation, we use $x_i \equiv x_i(\mathbf{b})$ and $x_j \equiv x_j(\mathbf{b})$. Accordingly, we use x_i^* and x_j^* for the bidders' allocations in the socially optimal outcome. Since j is the overwinner, $x_j \geqslant x_j^*$ and $x_j \geqslant x_i^* - x_i$. Consider a deviation $\mathbf{b}'_i$ of bidder i:

$$\mathbf{b}'_i = (\underbrace{v_i(x_i^*)/x_i^*, \ldots, v_i(x_i^*)/x_i^*}_{x_i^* \text{ marginal bids}}, 0, \ldots, 0),$$

towards obtaining $x_i^* - x_i \leqslant x_j$ additional units, that are held by bidder j under $\mathbf{b}$. First notice that the uniform bid of j under $\mathbf{b}$ cannot be larger than $v_j(x_j)/x_j$, because of our assumption of no-overbidding; thus, the uniform price $p(\mathbf{b}'_i, \mathbf{b}_j)$ is no larger than $v_j(x_j)/x_j$ as well. Assume for now that $v_i(x_i^*)/x_i^* > v_j(x_j)/x_j$, thus, i definitely obtains exactly x_i^* units under $(\mathbf{b}'_i, \mathbf{b}_j)$. Since $\mathbf{b}$ is a pure Nash equilibrium: $u_i(\mathbf{b}'_i, \mathbf{b}_j) \leqslant u_i(\mathbf{b}) \leqslant v_i(x_i)$, thus:

$$v_i(x_i^*) \leqslant v_i(x_i) + x_i^* \cdot p(\mathbf{b}'_i, \mathbf{b}_j) \leqslant v_i(x_i) + x_i^* \cdot (v_j(x_j)/x_j)$$

By solving for $v_j(x_j)$, we obtain:

$$v_j(x_j) \geqslant \frac{x_j \cdot (v_i(x_i^*) - v_i(x_i))}{x_i^*} \geqslant \frac{(x_i^* - x_i) \cdot (v_i(x_i^*) - v_i(x_i))}{x_i^*} \tag{8}$$

On the other hand, if $v_j(x_j)/x_j \geqslant v_i(x_i^*)/x_i^*$, then $v_j(x_j) \geqslant (x_j/x_i^*)v_i(x_i^*) \geqslant (x_j/x_i^*)(v_i(x_i^*) - v_i(x_i))$ and (8) remains valid. By the submodularity and the monotonicity of v_i, we also have:

$$v_i(x_i) \geqslant \frac{x_i}{x_i^*} v_i(x_i^*) \geqslant \frac{x_i}{x_i^*} v_i(x_i^* - x_i) \geqslant \frac{x_i}{x_i^*}(v_i(x_i^*) - v_i(x_i)) \tag{9}$$

Summing inequalities (8) and (9) yields: $v_i(x_i) + v_j(x_j) \geqslant v_i(x_i^*) - v_i(x_i)$. To this latter inequality, we add $v_j(x_j) \geqslant v_j(x_j^*)$ and $v_i(x_i)$ in both sides, to obtain the stated result. □

4 A Matching Lower Bound

We now present a lower bound construction for the standard bidding interface, establishing that our upper bound is tight, even for two bidders.

Theorem 2 *For any $k \geqslant 8$, the Price of Anarchy of pure Nash equilibria of the uniform price auction with submodular bidders is at least:*

$$1 + \frac{(1 - \frac{1}{k})(1 + \mathcal{W}_0(-e^{-2}))}{\frac{1}{k-1} + 1 + (-\mathcal{W}_0(-e^{-2})(1 - \frac{1}{k}))\ln(-\mathcal{W}_0(-e^{-2}) + \frac{1}{k})}$$

and approaches $(2 + \mathcal{W}_0(-e^{-2}))/(1 + \mathcal{W}_0(-e^{-2})) \approx 2.1885$ *as k grows.*

Proof We construct an instance of the Uniform Price Auction with two bidders and $k \geqslant 8$ units. Let $x \in \{1, 2, \ldots, k-2\}$ be a parameter that we will set later on. The valuation function of bidder 1 assigns value only for the first unit, which equals:

$$m_{11} = \frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i}$$

For the remaining units, we have $m_{1j} = 0$, for every $j \geqslant 2$. The valuation function of bidder 2 is given by the following marginal values:

$$m_{2j} = \begin{cases} 1, & 1 \leqslant j \leqslant k-1 \\ 0, & j = k \end{cases}$$

Hence, the optimal allocation is for bidder 1 to obtain only 1 unit and for bidder 2 to obtain $k-1$ units. We will now construct an equilibrium profile, where bidder 1 will end up winning much more than just a single unit, forcing bidder 2 to a lower total value, and resulting in a suboptimal allocation. To do this, we need to distribute the actual value m_{11} that bidder 1 has for the first unit, across several units with appropriate marginal bids. This should be done in such a way, so that bidder 2 will not have an incentive to change her strategy so as to obtain more units.

Based on these thoughts, we consider the following bidding profile $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2)$:

$$b_{1j} = \begin{cases} 1 - \frac{x}{k-1}, & j = 1 \\ 1 - \frac{x}{k-j+1}, & j = 2, \ldots, k-x \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad b_{2j} = \begin{cases} \epsilon, & j = 1, \ldots, x \\ 0, & j > x \end{cases}$$

Here, $\epsilon > 0$ can be any arbitrarily small positive quantity, no larger than 1.

Let us illustrate for $k = 11$, that this construction yields a better lower bound than the previously known bound of $2 - \frac{1}{k}$ of [7]. By setting $x = 2$, we obtain the following instantiation of the defined valuation functions and bidding vectors:

$$\begin{aligned} \mathbf{m}_1 &= \left(\frac{7487}{1260}, 0, 0, \ldots, 0, 0, 0\right), & \mathbf{b}_1 &= \left(\frac{8}{10}, \frac{8}{10}, \frac{7}{9}, \frac{6}{8}, \frac{5}{7}, \ldots, \frac{1}{3}, 0, 0\right) \\ \mathbf{m}_2 &= (1, 1, \ldots, 1, 1, 0), & \mathbf{b}_2 &= (\epsilon, \epsilon, 0, 0, \ldots, 0, 0) \end{aligned}$$

It can be verified that this profile already yields a lower bound of $2 + \frac{12}{1645}$ on the Price of Anarchy. Coming back to the analysis for general k and x, we will first ensure that both bidding vectors $\mathbf{b}_1, \mathbf{b}_2$ adhere to no-overbidding. For the vector $\mathbf{b}_1$, it suffices to note that

$$\sum_{j=1}^{k-x} b_{1j} = \frac{k-1-x}{k-1} + \sum_{j=2}^{k-x} \frac{k-j+1-x}{k-j+1} = \frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i} = m_{11}$$

where the last equality holds by changing indices and setting $i = k - j + 1 - x$. Therefore, we have that $\sum_{j=1}^{k-x} b_{1j} = v_1(k-x)$. And this implies directly that, for any $\ell < k - x$, we have $\sum_{j=1}^{\ell} b_{1j} < v_1(\ell)$. It is also straightforward that, for $\ell > k - x$, the no-overbidding assumption cannot be violated. Similarly, for the vector $\mathbf{b}_2$, it is easy to check that it complies to no-overbidding.

Under $\mathbf{b}$, bidder 1 obtains $k - x$ units and bidder 2 obtains x units. Notice that in this profile the uniform price is 0, as there is no contest for any unit; bidder 1 bids for exactly $k - x$ units, while bidder 2 bids for x units. All other bids are 0.

We now argue that $\mathbf{b}$ is a pure Nash equilibrium, under the assumption that whenever there is a tie in a deviation from $\mathbf{b}$, bidder 1 always gets the unit in question. Bidder 1 clearly has no incentive to deviate. She is interested only in the first unit, and there is no incentive to win more units for her. Note also that she cannot lose the first unit (even if she bids a zero vector) due to the tie breaking rule.

Let us examine the case of bidder 2. Since bidder 2 is not interested in the last unit, we can consider only deviation vectors $\mathbf{b}'_2$ with $b'_{2k} = 0$. Note that under $\mathbf{b}$, $u_2(\mathbf{b}) = x$. Hence, bidder 2 does not have an incentive to try to obtain less than x units, since the price will then still remain 0, and she will only have lower utility. It therefore suffices to consider what happens when she tries to obtain ℓ additional units, where $\ell = 1, \ldots, k - x - 1$. To do so, bidder 2 must outbid some of the winning bids of $\mathbf{b}_1$. In particular, to obtain ℓ additional units at the minimum possible price, she must outbid the bid b_{1t} of bidder 1, where t is the index $t = k - x - (\ell - 1)$. If she issues a bid $\mathbf{b}'_2$, where the first $x + \ell$ coordinates outbid b_{1t} and the remaining bids are 0, then she will obtain exactly $x + \ell$ units, and the new price (i.e., the new highest losing bid) will be precisely b_{1t}. However, any such attempt will grant bidder 2 utility equal to $u_2(\mathbf{b})$, since

$$\begin{aligned} u_2(\mathbf{b}_1, \mathbf{b}_2') &= v(x+\ell) - (x+\ell)\cdot b_{1t} \\ &= x+\ell-(x+\ell)\cdot(1-\frac{x}{x+\ell}) = x = u_2(\mathbf{b}). \end{aligned}$$

We conclude that the profile $\mathbf{b}$ is a pure Nash equilibrium. The ratio of the optimal social welfare to the one in $\mathbf{b}$ is at least:

$$\begin{aligned} &\frac{SW(\mathbf{x}^*)}{SW(\mathbf{b})} = \frac{v_1(1)+v_2(k-1)}{v_1(k-x)+v_2(x)} \\ &= 1+\frac{k-1-x}{\frac{k-1-x}{k-1}+\sum_{i=1}^{k-1-x}\frac{i}{x+i}+x} = 1+\frac{k-1-x}{\frac{k-1-x}{k-1}+k-1-x\sum_{i=x+1}^{k-1}\frac{1}{i}} \\ &\geqslant 1+\frac{k-1-x}{\frac{k-1-x}{k-1}+k-1-x\int_{x+1}^{k}\frac{1}{y}dy} \geqslant 1+\frac{k-1-x}{\frac{k-1-x}{k-1}+k-1-x\ln\frac{k}{x+1}} \end{aligned} \tag{10}$$

At this point we set $x = \lfloor -\mathcal{W}_0(-e^{-2})\dot{(}k-1)\rfloor$, where $\mathcal{W}_0$ is the first branch of the Lambert W function. To continue from (10), we will need to ensure that $-\mathcal{W}_0(-e^{-2})(k-1)-1>0$, which holds for $k \geqslant 8$. Continuing from (10), we have:

$$\begin{aligned} \frac{SW(\mathbf{x}^*)}{SW(\mathbf{b})} &\geqslant 1+\frac{k-1-\lfloor -\mathcal{W}_0(-e^{-2})(k-1)\rfloor}{\frac{k-1-\lfloor -\mathcal{W}_0(-e^{-2})(k-1)\rfloor}{k-1}+k-1-\lfloor -\mathcal{W}_0(-e^{-2})(k-1)\rfloor\ln\frac{k}{\lfloor -\mathcal{W}_0(-e^{-2})(k-1)\rfloor+1}} \\ &\geqslant 1+\frac{k-1-(-\mathcal{W}_0(-e^{-2})(k-1))}{\frac{k-(-\mathcal{W}_0(-e^{-2})(k-1))}{k-1}+k-1-(-\mathcal{W}_0(-e^{-2})(k-1)-1)\ln\frac{k}{-\mathcal{W}_0(-e^{-2})(k-1)+1}} \\ &\geqslant 1+\frac{k-1-(-\mathcal{W}_0(-e^{-2})(k-1))}{\frac{k}{k-1}+k-(-\mathcal{W}_0(-e^{-2})(k-1))\ln\frac{k}{-\mathcal{W}_0(-e^{-2})k+1}} \\ &= 1+\frac{(1-\frac{1}{k})(1+\mathcal{W}_0(-e^{-2}))}{\frac{1}{k-1}+1-(-\mathcal{W}_0(-e^{-2})(1-\frac{1}{k}))\ln(\frac{1}{-\mathcal{W}_0(-e^{-2})+\frac{1}{k}})} \\ &= 1+\frac{(1-\frac{1}{k})(1+\mathcal{W}_0(-e^{-2}))}{\frac{1}{k-1}+1+(-\mathcal{W}_0(-e^{-2})(1-\frac{1}{k}))\ln(-\mathcal{W}_0(-e^{-2})+\frac{1}{k})}. \end{aligned} \tag{11}$$

The first inequality is inferred using the well-known bounds of $\lfloor y\rfloor$, namely that $y \geqslant \lfloor y\rfloor \geqslant y-1$ for any $y \geqslant 0$. Moreover, notice that $(-\mathcal{W}_0(-e^{-2})(k-1)-1)>0$ since $k \geqslant 8$.

Let $f(k)$ be the right-hand side of (11). The theorem then follows by observing that:

$$\lim_{k\to\infty} f(k) = 1+\frac{1+\mathcal{W}_0(-e^{-2})}{1-\mathcal{W}_0(-e^{-2})\cdot\ln(-\mathcal{W}_0(-e^{-2}))} = \frac{2+\mathcal{W}_0(-e^{-2})}{1+\mathcal{W}_0(-e^{-2})},$$

where the last equality is derived by using the property that $\ln(-\mathcal{W}_0(y)) = -\mathcal{W}_0(y)+\ln(-y)$ for $y\in[-e^{-1},0)$, see [6]. □

5 Conclusions

We have presented a tight bound on the Price of Anarchy of pure Nash equilibria of the uniform price auction, for bidders with submodular valuation functions. Our results, together with the previous works on this topic exhibit that although such mechanisms are not truthful, they achieve good performance guarantees in terms of social welfare. This justifies and motivates their use in practical scenarios.

Regarding future research, we noted that our upper bound is also valid for the uniform bidding interface, where each bidder submits a single per-unit bid and an upper bound on the maximum number of units he wishes to receive. However, for the case of uniform bidding with 2 bidders, we showed a slightly better upper bound of 2; this, in combination with our lower bound of 2.1885 for 2 bidders under the standard bidding interface, yields a separation of the performance of the two formats for 2 bidders, that motivates further investigation of uniform bidding for more bidders. In addition to this, there are several even more intriguing open questions in multi-unit auctions. First, it is not clear to us if our proof can be recast into the smoothness framework of [16, 19]. Second, when moving beyond submodular valuation functions, to the superclass of subadditive functions, the known bounds are not tight. It still remains elusive to produce lower bounds tailored for subadditive functions, and the best known upper bound is 4 [7]. Finally, a major open problem is to tighten the known gaps for the set of Bayes-Nash equilibria.

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