Tight Welfare Guarantees for Pure Nash Equilibria of the Uniform Price Auction

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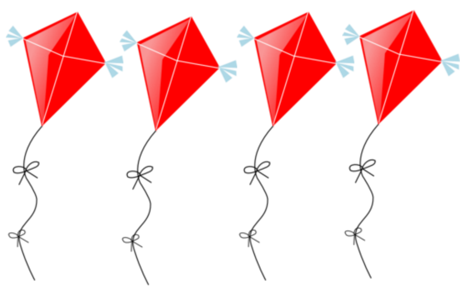
Uniform Price Auction

Allocate k units of an item to a set of n bidders (k highest marginal bids win). Charge each winner the *highest losing bid* per unit won.

v_1 = (7, 1, 1, 0)
b_1 = (5, 3, 1, 0)



- Uniform price = 2

v_2 = (10, 6, 1, 1)
b_2 = (6, 4, 2, 0)

- u_1 = 7+1 - 2*2 = 4
- u_2 = 10+6 - 2*2 = 12

Generalization of Second Price Auction [1] but *not Vickrey*. Employed in real-life settings such as bond auctions and online brokers.

Formally:

The auctioneer receives bids in one of the following two ways:

- ***standard bidding***: bidder i submits k-non increasing bids, i.e. $b_{i1} \geq b_{i2} \geq \cdots b_{ik}$
- ***uniform bidding***: bidder i submits a single per-unit bid and a quota

Given a profile $\boldsymbol{b} = (\boldsymbol{b}_1, \ \boldsymbol{b}_2, \dots, \boldsymbol{b}_n)$

- Allocation $\boldsymbol{x}(\boldsymbol{b}) = (x_1(\boldsymbol{b}), x_2(\boldsymbol{b}), \dots, x_n(\boldsymbol{b}))$ where: $x_i(\boldsymbol{b})$ = number of units bidder i is allocated
- Bidder i pays $x_i(\boldsymbol{b})$p($\boldsymbol{b}$) where p($\boldsymbol{b}$) is the highest losing bid.

Bidder i has a submodular valuation expressed as a non-increasing vector of *marginal* values, i.e. $\mathbf{v_i} = (m_{i1}, m_{i2}, \dots, m_{in})$ where:

m_{ij} = extra value derived by agent i for getting item j

Given a profile $\boldsymbol{b}$, the utility of bidder i is

$u_i(\boldsymbol{b}) = \sum_{j=1}^{x_i(\boldsymbol{b})} m_{ij} - x_i(\boldsymbol{b})p(\boldsymbol{b})$

Pure Nash Equilibria and No-Overbidding

A bidding profile $\mathbf{b} = (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n)$ is a Pure Nash Equilibrium if for every bidder i and every $\boldsymbol{b}_i{}'$:

$$u_i(\boldsymbol{b}) \geq u_i(b'_i, \boldsymbol{b}_{-i})$$

We assume bidders will not submit bids that may result in negative utility, i.e. for a given profile $\boldsymbol{b}$, every bidder i and any $l \leq k$ it holds

$$\sum_{j=1}^{l} b_{ij} \leq \sum_{j=1}^{l} m_{ij}$$

Inefficiency of Non-Overbidding Equilibria

Social Welfare: For a given profile $\mathbf{b}$ the Utilitarian Social Welfare is

$$SW(\boldsymbol{b}) = \sum_{i=1}^{n} \sum_{j=1}^{x_i(\boldsymbol{b})} m_{ij}$$

Price of Anarchy of no-overbidding Pure Nash Equilibria:

$$PoA = sup_{\boldsymbol{b}} \frac{SW(OPT)}{SW(\boldsymbol{b})}$$

In a Uniform Price Auction, bidders have incentives to shade their bids and these actions may result to equilibria. Here is an inefficient equilibrium:

Demand Reduction Effect [2]

v_1 = (1/3, 0, 0)
b_1 = (1/3, 0, 0)

v_2 = (1, 1, 1)
b_2 = (1, 1, 0)

Uniform price = 0

- $\mathbf{b}$ = (b_1, b_2) is an equilibrium
- Revealing true profile for bidder 2 results in a price that is too high for her!

SW(OPT) = 3
SW($\mathbf{b}$) = 2 + 1/3

$$\frac{SW(OPT)}{SW(\boldsymbol{b})} = \frac{9}{7} \approx 1.28$$

Can it get worse? Previously known [3] PoA lower-bound $2 - \frac{1}{k}$.

Main Result: The Price of Anarchy of non-overbidding pure Nash equilibria of the Uniform Price Auction with submodular bidders is

$$\frac{2 + \mathcal{W}_0(-e^{-2})}{1 + \mathcal{W}_0(-e^{-2})} \approx 2.188,$$

where $\mathcal{W}_0$ is the first branch of the Lambert function.

Upper Bound: Not a smoothness proof!

Lower Bound Construction: For k=*11* consider the profile

v_1 = (5.942, 0, 0, 0, 0, 0, 0, 0, 0 ,0 , 0)
b_1 = (8/10, 8/10, 7/9, 6/8, 5/7, 4/6, 3/5, 2/4, 1/3, 0, 0)

v_2 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0)
b_2 = (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)

- Bidder 1 bids an harmonic series of marginal bids that sum to $\sum_{j=1}^{8} \frac{j}{j+2} + \frac{8}{10} \approx 5.942$
- Should bidder 2 compete for more than 2 units, she will only introduce a uniform price which leaves her indifferent in terms of utility.
- SW(OPT) / SW(b) = 2.007
- For large values of k (>250) we approach 2.188 with this construction.

References

[1] M. Friedman. *A Program for Monetary Stability*. Fordham University Press, New York (1960)

[2] L. Ausubel, P. Cramton. *Demand reduction and inefficiency in multi-unit auctions.* Technical report, University of Maryland (2002).

[3] de Keijzer, B, Markakis, E., Schäfer, G., Telelis, O.: *Inefficiency of standard multi-unit auctions.* ESA 2013, Springer, Berlin. Extended version available as arXiv:1303.1646 (2013)