Partial Allocations in Budget-Feasible Mechanism Design: Bridging Multiple Levels of Service and Divisible Agents

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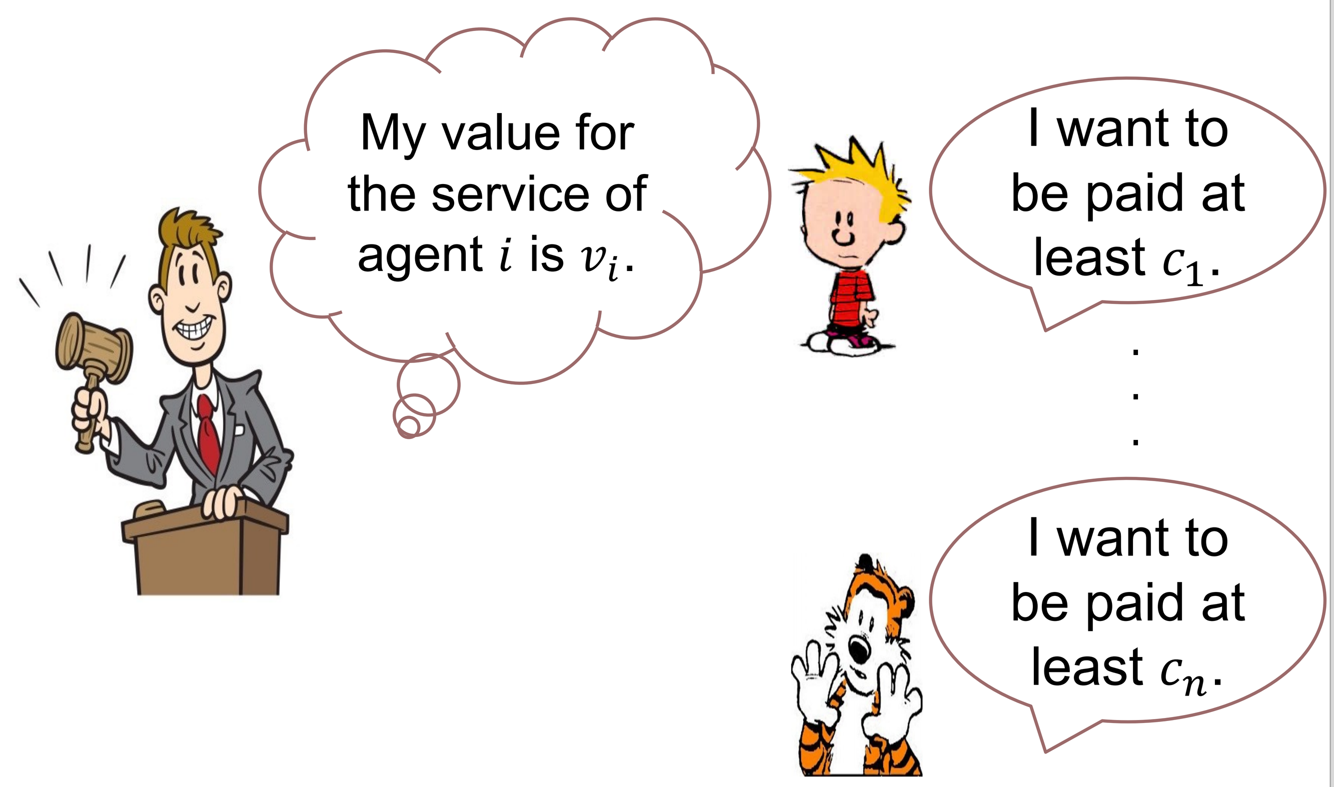
Budget Feasible Procurement Auctions

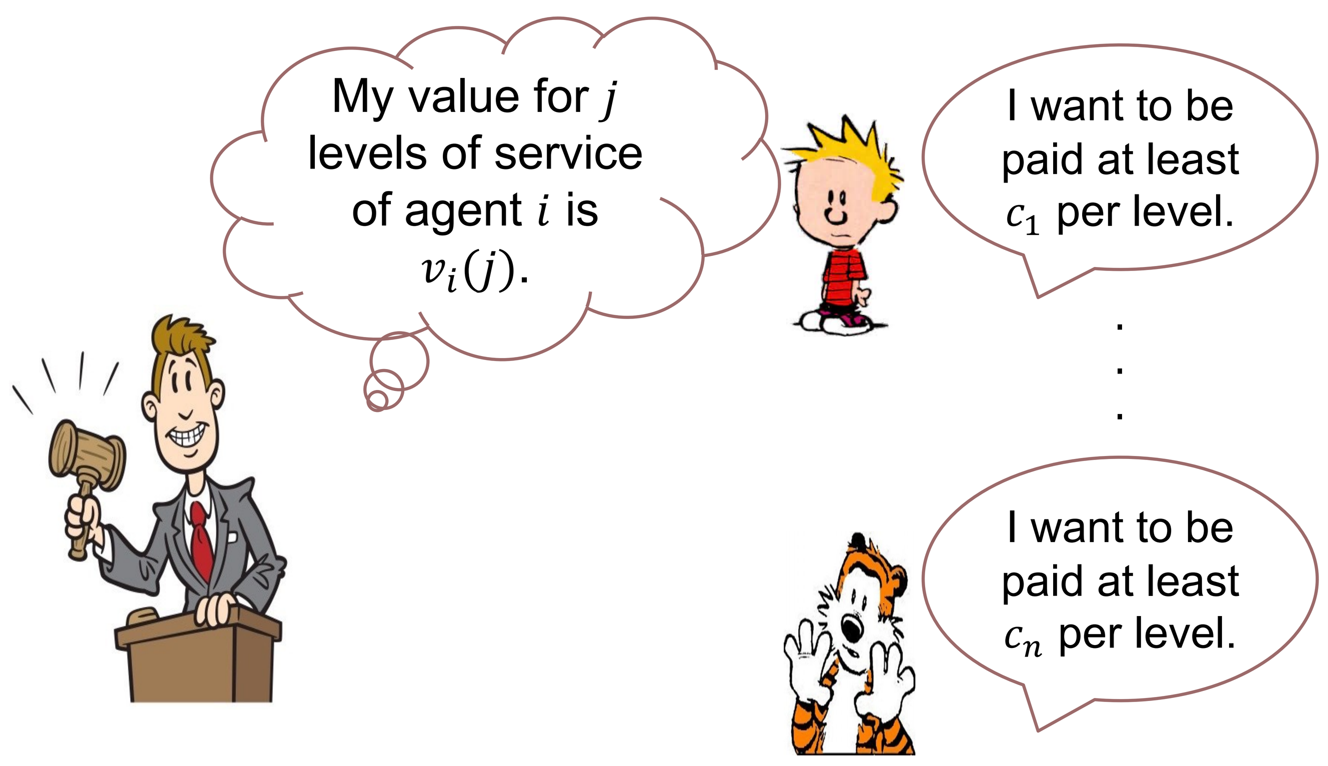
An auctioneer with a budget $B > 0$ is looking to hire a subset of n strategic agents.



Question [Singer '10]: **Who** should the auctioneer hire and how much should he **pay** (keeping in mind his budget B)?

Multiple Levels of Service Model (MLoS):

Agents can offer k levels of service.



Question (this work): What **hiring scheme** should the auctioneer implement and how much should he **pay** (keeping in mind his budget B)?

Formally:

An auctioneer with a budget B and a set of agents $N = \{1, \ldots, n\}$ with k levels of service.

For each $i \in \mathrm{N}$, a private cost parameter $c_i \geq 0$.

Bidders can be hired for some levels of service (e.g., we can think of a service having premium versions).

A mechanism $M = (\boldsymbol{x}, \boldsymbol{p})$ consists of:

1. An allocation algorithm that takes as input a vector $\mathbf{c} = (c_i)_{i \in N}$ of costs and outputs an allocation $\mathbf{x}(\boldsymbol{c}) \in \{0, \ldots, k\}^n$.
2. A payment rule that determines the payments $\boldsymbol{p}(\boldsymbol{c}) \in \mathbb{R}^n_{\geq 0}$ of the auctioneer.

- For each $i \in N$, there is a **non-decreasing concave** value function $v_i: \{0,1,\ldots,k\} \mapsto \mathbb{R}_{\geq 0}$.
- The total value of the auctioneer is $V(\boldsymbol{x}(\boldsymbol{c})) := \sum_{i=1}^{n} v_i(x_i(\boldsymbol{c}))$.
- For each $i \in N$ we assume that $c_i k \leq B$ (each bidder can be hired entirely on their own).

Truthful Budget Feasible Mechanisms for MLoS

The non-strategic optimal solution $OPT_I^k(\boldsymbol{c})$:

$$\max_{x \in \{0,\ldots,k\}^n} \sum_{i=1}^{n} v_i(x_i)$$

subject to

$$\sum_{i=1}^{n} c_i x_i \leq B$$

α-approximate mechanism: for every profile $\boldsymbol{c}$ it holds that

$$OPT_I^k(\boldsymbol{c}) \leq \alpha V(\boldsymbol{x}(\boldsymbol{c})).$$

Design Objectives for $M = (\boldsymbol{x}, \boldsymbol{p})$

Budget-feasibility: A payment rule should satisfy that $\forall \boldsymbol{c}$

$$\sum_{i=1}^{n} p_i(\boldsymbol{c}) \leq B.$$

Individual Rationality: Bidders should be incentivized to participate, i.e., $\forall \boldsymbol{c}$ and all $i \in N$

$$p_i(\boldsymbol{c}) \geq x_i(\boldsymbol{c}) c_i.$$

Truthfulness: Bidders have no incentives to lie about their costs.

Primary Goal of the Auctioneer: Find an α-approximate mechanism for the smallest α possible.

Truthful Mechanism Design for Single-parameter Domains

Definition: An allocation algorithm $\boldsymbol{x}$ is monotone if for every profile $\boldsymbol{c}$, every bidder $i \in N$ and every $c_i' \leq c_i$ it holds that $x_i(c_i', \boldsymbol{c}_{-i}) \geq x_i(\boldsymbol{c})$.

Myerson's Characterization: In a single parameter domain, a mechanism $M = (\boldsymbol{x}, \boldsymbol{p})$ is **truthful** and **individually rational** if and only if

1) $\boldsymbol{x}$ is monotone

2) $p_i(\boldsymbol{c}) = c_i x_i(\boldsymbol{c}) + \int_{c_i}^{\infty} x_i(y, \boldsymbol{c}_{-i}) dy, \ \forall i \in N.$

A Mechanism for Multiple Levels of Service

Input: A profile $\boldsymbol{c}$ and parameters α, β.

1. Let $i^* = argmax_{i \in N} \frac{v_i(k)}{OPT_F^k(\boldsymbol{c}_{-i})}$. ($OPT_F^k(\boldsymbol{c})$ is the fractional relaxation of $OPT_I^k(\boldsymbol{c})$.)
2. If $v_{i^*}(k) \geq \beta OPT_F^k(\boldsymbol{c}_{-i^*})$, then set $x_{i^*} = k$ and $x_i = 0$ for all $i \neq i^*$.
3. Else:
   1. Solve $OPT_F^k(\boldsymbol{c})$ and call its allocation $\boldsymbol{x}^*$ -> A list of decreasing marginal rates of the form (marginal value)/cost.
   2. Initialize $\boldsymbol{x}$ to be the integral part of $\boldsymbol{x}^*$.
   3. Keep removing the last element from the list and decrementing $\boldsymbol{x}$ until $\sum_{i=1}^{n} v_i(x_i) \geq \alpha OPT_F^k(\boldsymbol{c})$ holds minimally.
4. Return $\mathbf{x}$ and set $\mathbf{p}(\boldsymbol{c})$ according to Myerson.

Theorem 1: There exist constants α, β for which the mechanism is individually rational, truthful, budget-feasible and $(2 + \sqrt{3})$-approximate.

Natural Connection with the Divisible-Agents Scenario: Each agent $i \in N$ can be hired fractionally (e.g., think of hiring them for a fraction of their time).

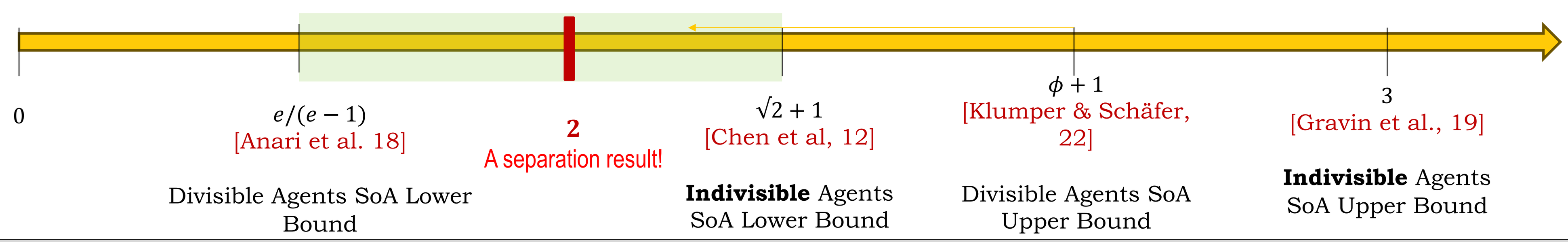
For each $i \in N$, there is a non-decreasing and concave function $\bar{v}_i: [0,1] \mapsto \mathbb{R}_{\geq 0}$.

⟐ L-regularity: For $L \geq 1$, a function $\bar{v}: [0,1] \mapsto \mathbb{R}_{\geq 0}$ is *L-regular Lipschitz* if $\bar{v}(x) \leq xL\, \bar{v}(1)$.

⟐ The above mechanism can used as a dicretization procedure for this problem, i.e., as $k \mapsto \infty$, we approach the divisible setting.

Theorem 2: There is a mechanism for Divisible Agents that is individually-rational, truthful, budget-feasible and $L(\phi + 1)$-approximate.

The Linear ($L = 1$) Case: The guarantee becomes $\phi + 1$, matching the state-of-the-art mechanism of [Klumper & Schäfer '22] for the divisible-agents scenario. **But we can do slightly better!**



References

[1] Singer, Y.: *Budget Feasible Mechanisms*. FOCS 2010

[2] Klumper, S., Schäfer, G.: *Budget Feasible Mechanisms for Procurement Auctions with Divisible Agents*. SAGT 2022

[3] Chen, N., Gravin, N., Lu, P.: *On the Approximability of Budget Feasible Mechanisms*. SODA 2011

[4] Gravin, N., Jin, Y., Lu, P., Zhang, C.: *Optimal Budget-feasible Mechanisms for Additive Valuations*. TEAC 2020

[5] Anari, N., Goel, G., Nikzad, A.: *Budget Feasible Procurement Auctions*. OR 2018