ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ
ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

ΣΧΟΛΗ ΕΠΙΣΤΗΜΩΝ & ΤΕΧΝΟΛΟΓΙΑΣ ΤΗΣ ΠΛΗΡΟΦΟΡΙΑΣ
SCHOOL OF INFORMATION SCIENCES & TECHNOLOGY

ΜΕΤΑΠΤΥΧΙΑΚΟ ΕΠΙΣΤΗΜΗ ΤΩΝ ΥΠΟΛΟΓΙΣΤΩΝ
MSc IN COMPUTER SCIENCE

Master of Science in Computer Science Thesis

MULTI-UNIT AUCTIONS

EQUILIBRIA AND INEFFICIENCY

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June 2017

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ABSTRACT

In this thesis we study the existence and (in)efficiency of Nash equilibria in multi-unit auctions, where an auctioneer is opting to sell several copies of a single indivisible good to a set of bidders. In the standard multi-unit auction format, each bidder submits a sequence of non-increasing marginal bids, for each additional unit i.e., a submodular curve. There are two dominant implementations that are being deployed, differing on the pricing scheme: the discriminatory price auction, in which each bidder pays the sum of her winning bids, and the uniform price auction, where the price for each unit is set to be the highest losing bid. Given the popularity of multi-unit auctions in practice, the study of such mechanisms has lately gained much attention in the literature. The main contribution of this thesis is a tight upper and lower bound on the inefficiency of pure Nash equilibria of the uniform price auction for bidders with submodular valuation functions, showing that the Price of Anarchy is bounded by 2.188. This resolves one of the open questions left open in previous works on multi-unit auctions. We complement our findings by providing an overview and further elaboration on the most recent results regarding the existence and inefficiency of Nash equilibria of these auction formats under full and incomplete information settings.

ACKNOWLEDGEMENTS

First and foremost, I would like to sincerely thank my advisor Dr. Vangelis Markakis for his help and guidance that made my first research experience very enjoyable. I was very fortunate to have him as my advisor and his support with my grad school applications was of great importance.

I am grateful to Dr. George D. Stamoulis and Dr. Orestis Telelis for their interest in this work and their helpful comments regarding the preparation of this manuscript and Alkmini Sgouritsa for our engaging discussions on multi-unit auctions.

During the past year, I've been privileged to be a member of the Theory Group of AUEB. I am particularly indebted to Georgios Amanatidis and Georgios Birmpas for our collaboration and their generous help. I will also miss the sweets and cookies which I took for granted every other Friday at our Theory Tea meetings!

I would like to thank my friends from AUEB: Ilias, Jason, Maria and Zannis, who all contributed to create and maintain a pleasant and stimulating learning environment. I wish them all the best.

Even if I never succeeded to fully explain what this thesis is about, I would finally like to warmly thank my family for their help in moments of doubt and Katerina for her love, affection and her willingness to memorize approximations of weird transcendental numbers.

My first contact with Theoretical Computer Science was due to Professor Ioannis Milis, at our undergraduate Algorithms course. I was fascinated by the subject and Professor Milis' teaching prowess certainly contributed to that. We were all deeply sad to hear about his passing. I will greatly miss our interactions and would like to dedicate this thesis to his memory.

CONTENTS

1 INTRODUCTION

OUTLINE

In this chapter we introduce the reader to auctions and in particular, auctions where copies of an indivisible items are sold to a set of bidders. We discuss the appeal of these auction formats in practice and state the goal and contribution of this thesis.

1.1 AUCTIONS AND GAME THEORY

In microeconomic theory, a market is defined as a variety of systems, institutions, procedures, social relations and infrastructures whereby parties engage in *exchange* of resources. We can identify two types of parties: buyers and sellers. When there exists a matching in terms of incentives for a particular resource; i.e. a buyer is interested in acquiring resource r while a seller is interested in trading r, there is an inherent conflict of interest. A buyer would be happy acquiring the resource in a reasonable (or rather low) price while a seller would be happy maximizing his individual profit by selling the resource at a high price. The outcome of each transaction highly depends on the strategic behavior that the parties exhibit during their interaction.

A specialized resource exchange market is a market that consists of one seller who is looking to allocate certain resources and many buyers that interact with the seller simultaneously (or almost simultaneously) and are willing to provide financial rewards to the seller to acquire these resources. This type of market is called an *auction*, while the seller is commonly referred to as the *auctioneer*.

Instances of auctions have been noted by historians at past centuries. In the fifth century B.C., Herodotus mentions auctions taking place in ancient Babylon. It is also known that auctions were taking place in ancient Athens and it can be said that they were quite popular in the Roman Empire and in medieval times. In the middle of the 18th century, reputable auction houses started to emerge, some of them being active to modern days. However, it is almost certain that the rule-making that was being used to define auctions in the previous centuries was not studied under any notion of scientific rigor until the last century and modern years. Still, there could be a possible explanation why these types of markets were standardized and widely popular: at an auction, the auctioneer feels safer having no competition from other sellers and is hoping that the multitude of bidders interested in resources will imply he sells for a high price. At the same time each of the buyers enjoys the fact that he has a say to the derivation of the price.

In modern times, auctions represent a non-negligible part of global economic activity. In 2008, the National Auctioneers Association reported that the gross revenue of the auction industry for that year was approximately 268.4 billion dollars, with the fastest growing sectors being agricultural, machinery, and equipment auctions and residential real estate auctions. Many auctions are also being administered by the State (e.g. spectrum auctions). Moreover, due to the World Wide Web, Online Auctions are widely implemented in practice. Popular auction platforms are eBay, eBid and Bonanza.

Auctions are regularly studied under the prism of game theory, since auctions can be represented as strategic games. Game theory is the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Since the field was established by John von Neumann and Oscar Morgenstern with the publication of the book *Games and Economic Behavior* in 1964, it has transformed microeconomics and is actively used in political science, psychology, biology and computer science. A landmark development in the field was the work of John F. Nash and in particular his conception of a solution concept for strategic games which is known as the Nash equilibrium.

Computer scientists are interested in several questions regarding Nash equilibria. One of those questions is the study of Nash equilibria under the lens of approximation. For instance, in an auction setting, researchers are interested in measuring how bad is the *social welfare* of an equilibrium compared to the optimal one. The formalization of this concept was conceived by Koutsoupias and Papadimitriou [1999] under the term *Price of Anarchy*.

1.2 MULTI-UNIT AUCTIONS

Imagine an auction where a single indivisible item is sold to a set of bidders. The auctioneer asks for bids simultaneously. Based on the bids received, there are two components the auction designer needs to specify in order to fully define the rules of the auction: (i) the winner of the auction (allocation rule) (ii) the pricing scheme to be implemented. In almost all reasonable auction types the winner of the auction is the issuer of the highest bid. There are several possible pricing rules the auctioneer implements (highest bid received, second highest bid received etc.).

The generalization of the auction of a single indivisible item is to auction several copies of this item, a *multi-unit auction*. In this setting, the bidders submit non-increasing marginal values for each unit. The three most prevailing auction formats are: the *Generalized Vickrey Auction*,the *Discriminatory Price Auction* and the *Uniform Price Auction*. All three formats share a common allocation rule: the highest marginal bids win and the issuing bidders are granted a unit per winning bid. Pricing schemes are, once again, the only difference. In the Generalized Vickrey Auction, being a generalization of the well known Second Price Auction of a single indivisible good, the bidders are charged according to the Clarke payment rule. This auction format was proposed and initially studied by Vickrey [1961]. In the Discriminatory Price Auction each bidder pays the sum of her winning bids, since this format is a natural generalization of the first price auction. Finally, in the Uniform Price Auction (Friedman [1960]) the highest losing bid is charged per allocated unit. Note all three formats share an identical bidding interface. It is the pricing scheme that changes.

The Generalized Vickrey Auction retains the nice properties of its single-item version. The equilibrium achieved at this auction is always the optimal outcome and the auction discourages bidders of strategic behavior. In fact, it can be proven that the optimal way to bid in such an auction for a bidder is to submit her true valuations for the units. Hence, this auction is *truthful* and it can be shown that it is economically efficient. Despite its strong theoretical guarantees, the main drawback of this auction format that hinders its implementation in real-world settings is the fact that its pricing rule is relatively complex.

The Discriminatory Price Auction and the Uniform Price Auction are not truthful auctions. Bidders may have incentives to state bids that are different from their true valuations. This strategic behavior may lead to equilibria that are inefficient. This inefficiency is amplified under settings of incomplete information (bidders are operating on beliefs about who their opponents are) or when their valuation functions are non-submodular (the auctioneer stills requires submodular bids).

Nevertheless, due to the fact that the inefficiency of these auction formats has been shown by recent results to be bounded by a small constant and the fact that the implemented pricing schemes are relatively simple, practitioners tend to implement these auctions in practice quite often. Multi-unit auctions have been implemented in various application domains such as the auctions offerred by the U.S. and U.K. Treasuries for selling bonds to investors. They are also being deployed in various platforms, including several online brokers Ockenfels et al. [2006]; Milgrom [2004].

As Milgrom [2004] notes, the resurgence of interest in auction theory owes much to recent large-scale auctions which were designed under suggestions of experts. He adds that, since the US spectrum auctions of 1994 many implemented auctions have been, essentially, Uniform Price Auctions. The Uniform Price Auction is a particularly appealing auction format for practitioners since it by design promotes the *law of one price*, according to which identical goods have identical prices. Many bidders dislike price variation when identical goods are being sold.

The main contribution of this thesis is the derivation of tight welfare guarantees for the Uniform Price Auction in pure Nash equilibria and bidders with submodular valuation functions. This work will soon appear in Birmpas et al. [2017]. Additionally, we elaborate on recent results on the existence and inefficiency of Nash equilibria in full and incomplete information settings.

The rest of the thesis is structured as follows. In chapter 2 we provide some background on fundamental game theoretic notions with an emphasis on Nash Equilibria along with the model and key definitions of multi-unit auctions along with related work. In chapters 3 and 4 we study the inefficiency of the Discriminatory Price auction and the Uniform Price Auction respectively. Finally, in chapter 5 we mention interesting open problems in multi-unit auctions.

2 BACKGROUND, DEFINITIONS AND RELATED WORK

OUTLINE

In this chapter we provide some background on fundamental game theoretic notions with an emphasis on Nash Equilibria along with the model and key definitions of multi-unit auctions. In addition, we discuss related work.

2.1 STRATEGIC GAMES AND NASH EQUILIBRIA

We start by giving definitions of classes of Nash equilibria under the full information setting and the incomplete information setting of Harsanyi [1968].

2.1.1 *One-shot Simultaneous Move Games*

We begin with the definition of a *one-shot simultaneous move game*. Let $\mathcal{N}$ be a set of n players, $\mathcal{N} = \{1, \ldots, n\}$. Every player $i \in \mathcal{N}$ has her own *set of possible strategies* $\mathcal{S}_i$. All players select a strategy $s_i \in \mathcal{S}_i$ simultaneously. We denote the *vector of strategies* selected by each of the players as $\mathbf{s} = (s_1, \ldots, s_n)$. Such a vector will often be referred to as a *strategy profile*. Additionally, we denote the set of all possible ways in which players can pick strategies as $\mathcal{S} = \times_i \mathcal{S}_i$.

Every vector of strategies $\mathbf{s} \in \mathcal{S}$ determines different outcomes for different players. The final component we need to define a one-shot simultaneous move game is a *preference ordering* over the outcomes, which can be represented by a *utility function* (payoff function) $u_i : \mathcal{S} \mapsto \mathcal{R}$, which assigns a value to each outcome. A high value of $u_i(\mathbf{s})$ signifies that the outcome $\mathbf{s}$ is highly desirable for player i. Moreover, the fact that the utility of a player is a function of $\mathbf{s}$, and not simply of s_i, implies that the utility of a player depends not only on her own strategy but also on the strategies chosen by other players.

Definition 2.1 (cf. Nisan et al. [2007])**.** *A one-shot simultaneous move game consists of:*

- *A set of n players $\mathcal{N} = \{1, \ldots, n\}$*
- *For each player $i \in \mathcal{N}$, a set of possible strategies $\mathcal{S}_i$*
- *For each player $i \in \mathcal{N}$, a utility function $u_i : \mathcal{S} \mapsto \mathcal{R}$.*

The sum of utilities of all players that is achieved at a strategy profile $\mathbf{s}$ is called the *Utilitarian Social Welfare* of this strategy profile; i.e.

$$SW(\mathbf{s}) = \sum_{i=1}^{n} u_i(\mathbf{s}). \tag{2.1}$$

There are other definitions capturing richer aspects of strategic games. However, they are not the focus of our work.

Example 2.1 (Prisoner's Dilemma)**.** The classical example of a one-shot simultaneous move game is the Prisoners' dilemma. Imagine there are two prisoners on trial for a crime and each one of them has two choices: to confess the crime or remain silent. If both prisoners remain silent, the charges against them cannot be proven by the authorities and both will serve a short prison term, say 2 years. If only one of them confesses, her term is reduced to 1 year while the term of the other prisoner will be prolonged to 5 years (the first prisoner acts as a witness!). Finally, if both prisoners confess they will have to serve 4 years of prison each (authorities give them a break for cooperating).

We can model the situation described as a one-shot simultaneous move game. We can succinctly summarize the utilities along with the available strategies (confess, silent) of the two prisoners in a *utility matrix*.

	Confess	Silent
Confess	-2,-2	-1,-5
Silent	-5,-1	-4,-4

Table 2.1: Prisoners' Dilemma: Utility Matrix

2.1.2 *Nash Equilibria under Complete Information*

In example 2.1 we notice that when both prisoners confess, it is a *stable solution*. That is, if both prisoners confess and we ask each prisoner to change his action *unilaterally* (the other prisoner will still confess) the utility of the prisoner in question can only get lower (from -2 to -5). Therefore, since both bidders are rational, they would not choose to change their strategy. This stable solution is called in the literature a *pure Nash Equilibrium*. We now present the formal definition of this game-theoretic solution concept.

Definition 2.2 (cf. Osborne [2004])**.** *In a strategic game of n players, a vector of pure (deterministic) strategies $\mathbf{s}^*$ is a pure Nash Equilibrium when, for every player $i = 1, \dots, n$, it holds that $u_i(\mathbf{s}^*) \geq u_i(s'_i, \mathbf{s}^*{}_{-i})$, for every unilateral deviation $s'_i \in \mathcal{S}_i$.*

Fact 2.1. *There are strategic games with no Pure Nash Equilibria.*

We will demonstrate this fact in the example that follows.

Example 2.2 (Matching Pennies)**.** Imagine that two players called Even and Odd have a penny and must secretly turn the penny to heads or tails. The players reveal their choices simultaneously. If the revealed pennies match, then Even keeps both pennies (+1 for Even, -1 for Odd). Conversely, if the pennies do not match, player Odd keeps both pennies (-1 for Even, +1 for Odd). Since this is a two-player game with a finite number of strategies (heads, tails), we can once again model it with a utility matrix.

	Heads	Tails
Heads	1,-1	-1, 1
Tails	-1, 1	1,-1

Table 2.2: Matching Pennies: Utility Matrix

Notice that in every possible combination of strategies in this strategic game, there is one player (the one with a negative utility) with an incentive to deviate unilaterally and obtain a positive utility. Therefore, this game possesses no pure Nash Equilibria.

In pure strategy profiles, each player picks a chosen strategy deterministically. A natural extension for the definition of a strategic game is to allow players to *randomize* or pick a strategy s_i from a probability distribution $\mathcal{D}_i$ over the player's set of possible strategies $\mathcal{S}_i$. Naturally, we represent the product distribution of all players' strategies in a game as $\mathbf{D} = \times_i \mathcal{D}_i$. In this enhanced model, each player $i \in \mathcal{N}$ seeks to maximize her *expected utility* $\mathbb{E}_{\mathbf{s} \sim \mathbf{D}}[u_i(\mathbf{s})]$.

A stable solution in this generic model is called a *mixed Nash Equilibrium*.

Definition 2.3 (cf. Osborne [2004])**.** *In a strategic game of n players, a vector of mixed strategies $\mathbf{s}^* \sim \mathcal{D}$ is a mixed Nash Equilibrium when, for every player $i = 1, \dots, n$, it holds that*

$$\mathbb{E}_{\mathbf{s}^* \sim \boldsymbol{D}}[u_i(\mathbf{s}^*)] \geq \mathbb{E}_{\mathbf{s}^*_{-i} \sim \boldsymbol{D}_{-i}}[u_i(s'_i, \mathbf{s}^*_{-i})],$$

for every unilateral deviation $s'_i \in \mathcal{S}_i$.

As we mentioned above, in example 2.2 there are no pure Nash Equilibria. However, when players are allowed to randomize, we can identify the following mixed Nash Equilibrium: the strategy profile when each player picks each of his two strategies with probability 1/2. Then, the expected utility of each player is 0 and neither of them can improve by deviating unilaterally.

In fact, there is a much stronger statement concerning the existence of mixed Nash Equilibria due to Nash (1951) expressed with the following celebrated theorem.

Theorem 2.1 (cf. Nash [1951], Theorem 1)**.** *Any game with a finite set of players and finite set of strategies has a Nash equilibrium of mixed strategies.*

Theorem 2.1 is a fundamental contribution to game theory and one of the main reasons why Nash Equilibria have practical and philosophical importance in any setting where strategic agents interact with great implications to economics, computer science, biology and other sciences.

2.1.3 *Nash Equilibria under Incomplete Information*

So far we have assumed that players participating in a strategic game have full information about the utilities and strategies of all other players. However, in a real world setting, this is not usually the case. Players only have limited information that can also be called *beliefs* and therefore, try to pick their best strategies given the information available. These types of strategic games are named in the literature as *Bayesian Games* and the relevant solution concept used to analyze these games *Bayes-Nash Equilibrium*.

More formally, in games of incomplete information, each player $i \in \mathcal{N}$ has a private *type* $t_i \in T_i$, T_i being the set of possible types for player i drawn from a probability distribution π_i. The vector $\pi = (\pi_1, \ldots, \pi_n)$ is assumed to be drawn from a publicly known probability distribution π. [1] The utility of player $i \in \mathcal{N}$ is a function of her t_i in addition to the strategy profile $\mathbf{s}$ picked by all players.

This model of incomplete information was formalized mostly by Harsanyi [1968] and others in the 1960s and 1970s and is the standard working assumption in microeconomics literature when reasoning about incomplete information.

The Bayes-Nash equilibrium is a further generalization of the definition of the mixed Nash equilibrium.

Definition 2.4 (cf. Harsanyi [1968])**.** *A strategy of a player $i \in \mathcal{N}$ is a function $s_i : T_i \mapsto \mathcal{S}_i$. A strategy profile $\mathbf{s}$ is a Bayes-Nash equilibrium if for every player i and every type t_i it holds that*

$$\mathbb{E}_{t_{-i} \sim \pi_{-i}}[u_i(t_i, s_i(t_i), s_{-i}(t_{-i}))] \geq \mathbb{E}_{t_{-i} \sim \pi_{-i}}[u_i(t_i, s_i', s_{-i}(t_{-i}))],$$

for every unilateral deviation $s_i' \in \mathcal{S}_i$, where π_{-i} is the product distribution of t_{-i}.

Remark 2.1. *If we regard the full information setting as a special case of the Bayesian setting we can notice the following hierarchy of equilibria classes:*

Pure NE $\subseteq$ mixed NE $\subseteq$ Bayes NE.

1 Assume that there exists a special entity in the game called *Nature*. Nature assign types to each one of the n players. A reader accustomed to Bayesian probability notions may infer that the distributions π_ℓ, for $\ell = 1, \ldots, n$ are the *prior* distributions. The player $i \in \mathcal{N}$ only knows her own distribution π_i but not the types t_{-i}. Rather, she knows the prior distributions π_{-i} from which the types t_{-i} are drawn from.

2.2 QUANTIFYING THE (IN)EFFICIENCY OF EQUILIBRIA

The social welfare (equation (2.1)) of a strategy profile can be regarded as its measure of quality. Under this measure, equilibrium profiles may be far from the optimal strategy profile (the social optimum). The main reason this can be the case is the definitive lack of coordination that players in a strategic game have, due to the fact that each one of them is looking to maximize a different objective function. Furthermore, there are cases when players can only achieve the social optimum, when a centralized coordination mechanism chooses their strategies for them [2].

How bad is this lack of coordination for players? In other words, what is the worst social welfare achieved at one of the equilibria of a game compared to the social optimum? Notions of worst-case analysis are ever-present in the design and analysis of algorithms. For instance, we are interested in performance guarantees of approximation algorithms, the competitive ratio of online algorithms etc. The relevant notion for welfare of equilibria was first introduces by Koutsoupias and Papadimitriou [1999] as the *coordination ratio* although the term that dominated in subsequent works is the term *Price of Anarchy*. We present the definition for pure equilibria below.

Definition 2.5 (Koutsoupias and Papadimitriou [1999], Koutsoupias and Papadimitriou [2009], Papadimitriou [2001])**.** *Let $\mathcal{S}^*$ be the set of pure equilibria of a strategic game and OPT be the outcome that maximizes the social welfare. The Price of Anarchy of this strategic game is*

$$\sup_{\mathbf{s}\in\mathcal{S}^*} \frac{SW(OPT)}{SW(\mathbf{s})}.$$

We defer the presentation of definitions for more general equilibrium classes to subsection 2.4.3 tailored for the model of multi-unit auctions.

Deriving upper and lower bounds on the price of anarchy of strategic games has inspired interesting new mathematics and has led researchers in evaluating the performance of existing games and the design of new ones with improved welfare guarantees.

2.3 SINGLE ITEM AUCTIONS

Setting the rules of a game so that its solution concepts have several desirable properties is the objective of mechanism design theory. Regarding auctions, the rules of an auction (or an *auction format*) are the specification of an auction *mechanism*.

Definition 2.6. *Let $\mathcal{S}$ be the set of all possible strategy profiles and $\mathcal{X}$ be the set of possible allocations of an auction. A mechanism is a pair $\mathcal{M} = (f, P)$, where $f : \mathcal{S} \mapsto \mathcal{X}$ is the allocation algorithm and $P : \mathcal{S} \mapsto R^n$ is the pricing scheme.*

2 The social optimum is not necessarily an equilibrium.

The allocation algorithm intuitively describes which bidders gets which item(s), whereas the pricing scheme describes the payments of all participants.

Imagine an auction with a single indivisible item for sale. Let $\mathcal{N}$ be a set of n bidders, $\mathcal{N} = \{1, \ldots, n\}$. Each bidder has a private *valuation*; it represents how much a bidder is willing to pay. We denote the valuation of bidder $i \in \mathcal{N}$ for the item as $v_i \in R^+$.

The mechanism $\mathcal{M}$ (or auctioneer) accepts a vector of simultaneous non-negative *bids* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ from all bidders in $\mathcal{N}$ and grants the item to one bidder according to the allocation algorithm $f(\mathbf{b})$. Moreover, the auctioneer expects a *payment* $P_i(\mathbf{b})$ from each of the participants that is specified according to the chosen pricing scheme $P(\cdot)$. At a bidding profile $\mathbf{b}$, the utility achieved by each bidder i is a *quasi-linear* function of her private valuation v_i and the payment $P_i(\mathbf{b})$; i.e.

$$u_i(\mathbf{b}) = v_i - P_i(\mathbf{b}).$$

For single-item auctions, almost all reasonable mechanisms have the same allocation rule: assign the item to the issuer of the highest bid $w = \arg\max_{i \in \mathcal{N}} b_i$. Hence, these mechanisms differ in the choice of the implemented pricing scheme.

An important property that is sought in auction mechanisms is *truthfulness*.

Definition 2.7. *Let* $\mathbf{b}^* = (v_1, v_2, \ldots, v_n)$. *An auction mechanism is truthful when for every* $\mathbf{b}' \neq \mathbf{b}^*$ *and for every* $i \in \mathcal{N}$ *it holds that*

$$v_i(\mathbf{b}^*) - P_i(\mathbf{b}^*) \leq v_i(\mathbf{b}') - P_i(\mathbf{b}').$$

This definition implies that under a truthful mechanism the best strategy for every bidder is to reveal her truthful valuation. In such a mechanism, a rational bidder will not consider any strategic behavior; telling the truth is a dominant strategy.

We present two mechanisms for the single item auction: the First Price Auction and the Second Price auction. We discuss the impact of different bidding strategies in each case.

(a) **First Price Auction** The most standard and intuitive pricing scheme in a single item auction is for the winner of the auction to pay her declared bid, while others pay 0. What is the optimal strategy for a bidder in this setting?

Fix a bidder $i \in \mathcal{N}$. Bidder i has no incentive to bid above her v_i; should she win, this would only result in a negative utility. Moreover, bidding exactly v_i always achieves a utility of 0. What is the *best response* of bidder i to the bids of others that is between 0 and v_i? An individual rational bidder that seeks to maximize her utility in this setting would want to win and pay as less as possible. Let $b_i^* = \max_{\ell \in \mathcal{N}-i} b_\ell$. The best response function of i to bidding vector $\mathbf{b}_{-i}$ when bidding between 0 and v_i is $b_i^* + \epsilon$[3], where epsilon is an arbitrarily small positive quantity. In the full information setting,

3 We assume ties are not in favor of i.

when bidders are allowed to deviate unilaterally it is not hard see that all bidders have an incentive to change their bids. The winner has an incentive to lower her bids as much as possible and remain a winner (and increase their utility by paying less) while others have an incentive increase their bids (to potentially increase their utility by winning the auction[4]) to become winners. Therefore, since it is guaranteed that there cannot be a single best response for some bidders, in this type of games, Pure Equilibria are not guaranteed to exist. Additionally, in accordance to definition 2.7 this mechanism is not truthful since bidding a $b_i < v_i$ could be a profitable strategy.

(b) **Second Price Auction** In this type of auction the winner $w \in \mathcal{N}$ pays the *second* highest bid submitted to the mechanism. This mechanism has an important property: the price a bidder pays depends solely on the bids of others. Therefore, it is impossible for a bidder to manipulate the mechanism acting strategically and the best individual course of action is to tell the truth; i.e. for every bidder $i \in \mathcal{N}$, $b_i = v_i$. Hence, according to the definition 2.7, this is a truthful mechanism. Furthermore, the profile $\mathbf{b}^* = (v_1, v_2, \ldots, v_n)$ is a pure Nash equilibrium, since it maximizes the individual utility of every bidder.

2.4 MULTI-UNIT AUCTIONS

A generalization of single item auctions when bidding for more than one units of an item are multi-unit auctions, the main topic of this thesis. In this section we formally define the model used in the literature to describe them. We follow the notation from de Keijzer et al. [2013] and Markakis and Telelis [2015]. By default, we are focusing on the full information setting. We provide generalized definitions for the incomplete information setting in section 2.4.5.

We consider a multi-unit auction, involving the allocation of k units of a single item, to a set $\mathcal{N}$ of n bidders, $\mathcal{N} = \{1, \ldots, n\}$. Each bidder $i \in \mathcal{N}$ has a private valuation function $v_i : \{0, 1, \ldots, k\} \mapsto \mathbb{R}^+$, defined over the quantity of units that she receives, with $v_i(0) = 0$.

2.4.1 *Valuation Functions*

We consider models where bidders have *submodular* and *subadditive* valuation functions. A valuation function can also be specified through a sequence of *marginal values*, corresponding to the value that each additional unit yields for the bidder. For the j-th additional unit, the bidder obtains marginal value $v_i(j) - v_i(j-1)$, which we denote by m_{ij}. Then, the function v_i can be determined by the vector $\mathbf{m}_i = (m_{i1}, \ldots, m_{ik})$.

Definition 2.8. *A valuation function $f : \{0, 1, \ldots, k\} \mapsto \mathbb{R}^+$ is called submodular if for every $x < y$, $f(x) - f(x-1) \geq f(y) - f(y-1)$.*

By definition, for submodular functions it holds that $m_{i1} \geq \cdots \geq m_{ik}$.

4 Unless of course they have to overcome their willingness to pay v_i.

A submodular function can be considered as the discrete along of a continuous convex function. It is a suitable function when modeling processes with *diminishing returns* or decreasing marginal values. A bidder with a submodular valuation could be interested for many units of an item. However, her willingness to pay per item gradually diminishes as the number of units increases. Submodular functions also have the following well-known property. We present its proof for completeness.

Proposition 2.2. *Given $x, y \in \{0, 1, \dots, k\}$ with $x \leq y$, any non-decreasing submodular function f, with $f(0) = 0$, satisfies $yf(x) \geq xf(y)$. Moreover, when $x < y$, for any $j = 1, \dots, y - x$ the function f satisfies: $(f(x+j) - f(x))/j \geq (f(y) - f(x))/(y - x)$.*

Proof. Consider $x, y \in \{0, 1, \dots, k\}$ with $y \geq x$. When $x = 0$, the first statement of the proposition holds, for a non-decreasing submodular function f with $f(0) = 0$, because $yf(x) = xf(y) = 0$. When $x \geq 1$, we can express $xf(y)$ as follows, by using the marginal values of f:

$$
\begin{aligned}
xf(y) &= x\left(f(x) + \sum_{\ell=x+1}^{y} m_\ell\right) = xf(x) + x\sum_{\ell=x+1}^{y} m_\ell \\
&\leq xf(x) + x(y-x)m_x \leq xf(x) + (y-x)x\frac{f(x)}{x} = yf(x)
\end{aligned}
$$

The first inequality above is due to the non-increasing marginal values, i.e., that $m_x \geq m_\ell$, for $\ell = x+1, \dots, y$. The second inequality is justified by the fact that $m_x \leq m_\ell$ for all $\ell = 1, \dots, \ell$, thus, $m_x \leq f(x)/x$, which is the average of these marginal values.

For the second statement of the proposition, consider $y > x$ and $j = 1, \dots, y - x$. Define the function $g(j) = f(x+j) - f(x)$, over $\{1, \dots, y - x\}$. Using the fact that f is submodular non-decreasing, it can be straightforwardly verified that g is submodular non-decreasing as well, by Definition 2.8. Then it satisfies the first statement of the proposition, i.e., $(y-x)g(j) \geq jg(y-x)$, for any $j = 1, \dots, y - x$. Since $j \geq 1$ and $y - x \geq 1$, the second statement is implied for the function f. $\square$

Definition 2.9. *A valuation function $f : \{0, 1, \dots, k\} \mapsto \mathbb{R}^+$ is called subadditive if for every x, y, $f(x+y) \leq f(x) + f(y)$.*

Bidders with subadditive valuation functions do not necessarily have a decreasing willingness to pay. A subadditive bidder may be particularly interested in non-consecutive units of the item under the constraint posed by the definition. In the seminal work of Lehmann et al. [2006] it is shown that the class of submodular functions is strictly contained in the class of subadditive functions. We refer the reader to this work for more information regarding subadditive and other definitions an properties of several valuation functions.

2.4.2 *Bidding Interfaces and Allocation Algorithm*

To model a multi-unit auction, one needs to specify the format of bids the bidders submit to the auctioneer. In auction theory, (see Milgrom [2004]) there are two dominant formats: the *standard format* and the *uniform bidding format*.

In the standard format, each bidder $i \in \mathcal{N}$ submits a vector of k non-negative non-decreasing *marginal bids* $\mathbf{b}_i = (b_{i1}, \dots, b_{ik})$ with $b_{i1} \geq \cdots \geq b_{ik}$ (it is equivalent to say that each bidder submits a submodular bidding vector). We may refer to $\mathbf{b}_i$ as *bids*.

In the uniform bidding format, each bidder $i \in \mathcal{N}$ submits a single bid $\bar{b}_i$ along with a quantity $q_i \leq k$. This means that bidder i is willing to pay at most $\bar{b}_i$ per unit for up to q_i units (not willing to pay anything for $k - q_i$ units).

Bidders may submit and consider mixed strategies. Under full information, we define a probability distribution on each bidding vector $\mathbf{b}_i$ and we denote it by B_i. We then say that $\mathbf{b}_i \sim B_i$ or that $\mathbf{b}_i$ is drawn from probability distribution B_i. We also denote the as B the product distribution of all B_i. Consequently, we say that $\mathbf{b} \sim B$.

The auction implementer must bear in mind the trade-off that occurs between each of the two bidding interfaces when choosing the one that is most suitable. The standard bidding format is much more expressive for the bidders participating in the auction and offers them a greater flexibility. On the other hand, while the uniform bidding interface is more restrictive in terms of expressiveness, it is very simple in conception. Actually, as mentioned by Milgrom [2004] the uniform bidding interface is much closer to real-world implementations of multi-unit auctions.

The auctioneer receives bids from all bidders in the specified format[5] and runs an allocation algorithm that produces the *allocation* $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), x_2(\mathbf{b}), \dots, x_n(\mathbf{b}))$. The allocation algorithm is an instantiation of the algorithm for maximizing submodular functions of Nemhauser et al. [1978].

Allocation Algorithm (Markakis and Telelis [2015])

(a) **Set** $x_i = 0$, **for** $i = 1, \dots, n$.

(b) **For** $j = 1, \dots, k$ **do:**

   i. $i^* \leftarrow \arg\max_i b_i(x_i + 1)$

   ii. $x_{i^*} \leftarrow x_{i^*} + 1$

(c) **return x**

5 We can assume each bidder $i \in \mathcal{N}$ submits a vector $\mathbf{b}_i$ under both bidding interfaces. Under the uniform bidding format, we can denote the uniform bidding vector as $\mathbf{b}_i = (\underbrace{\bar{b}_{i1}, \dots, \bar{b}_{iq_i}}_{q_i \text{ bids}}, 0, \dots, 0)$, where $\bar{b}_i$ is the uniform bid and q_i the maximum quantity of units bidder i is competing for.

2.4.3 *Social Welfare and Price of Anarchy*

As in the single item auction model, the utility of every bidder $i \in \mathcal{N}$ is quasi-linear. In the multi-item auction model, the utility of i at a profile $\mathbf{b}$ additionally depends on her allocation $x_i(\mathbf{b})$. That is, for every bidder $i \in \mathcal{N}$, we have

$$u_i(x_i(\mathbf{b})) = v_i(x_i(\mathbf{b})) - P_i(x_i(\mathbf{b})).$$

The (utilitarian) *Social Welfare* achieved by the auction under a bidding profile $\mathbf{b}$ is defined as the sum of utilities of all interacting parties, inclusively of the auctioneer's revenue. This sum equals the sum of the bidders' values for their allocations[6]:

$$SW(\mathbf{b}) = \sum_{i=1}^{n} v_i(x_i(\mathbf{b}))$$

Our goal is to derive upper and lower bounds on the Price of Anarchy (PoA) of Nash equilibria of the Uniform Price Auction. This is the worst-case ratio of the optimal welfare, over the welfare achieved. If $\mathbf{x}^*$ denotes an optimal allocation, then, for the class of pure Nash Equilibria

$$PoA = \sup_{\mathbf{b}} \frac{SW(\mathbf{x}^*)}{SW(\mathbf{b})}$$

where the supremum is taken over pure equilibrium profiles. When considering equilibria under profiles of mixed strategies $\mathbf{b} \sim B$,

$$PoA = \sup_{B} \frac{SW(\mathbf{x}^*)}{\mathbb{E}_{\mathbf{b}\sim B}[SW(\mathbf{b})]}$$

where the supremum is taken over all possible distributions of bidding strategies.

Finally, following previous works on equilibrium analysis of auctions, e.g., Christodoulou et al. [2012]; Bhawalkar and Roughgarden [2011]; Markakis and Telelis [2015], we focus on *non-overbidding* equilibrium profiles $\mathbf{b}$, wherein no bidder ever outbids her value, for any number of units. That is, for any $\ell \leq k$, we assume

$$\sum_{j=1}^{\ell} b_{ij} \leq v_i(\ell). \tag{2.2}$$

Note that, this *does not* necessarily imply $b_{ij} \leq m_{ij}$, except for when $j = 1$: i.e., $b_{i1} \leq m_{i1} = v_i(1)$. In our analysis, we refer to non-overbidding vectors, $\mathbf{b}_i$, and profiles, $\mathbf{b}$, as *feasible*.

6 The utility of the auctioneer is $\sum_{i \in \mathcal{N}} P_i(\mathbf{b})$.

2.4.4 *Pricing Rules*

Recall that $P_i(x_i(\mathbf{b}))$ is how much a bidder has to pay to obtain the $x_i(\mathbf{b})$ allocated units. We will sometimes refer to a payment rule as a *pricing rule*. We consider pricing rules that are *discriminatory price dominated.*

Definition 2.10. *A pricing rule is discriminatory price dominated when under any profile* $\mathbf{b}$ *and for every bidder* $i \in \mathcal{N}$ *it holds that*

$$P_i(x_i(\mathbf{b})) \leq \sum_{j \in [x_i(\mathbf{b})]} b_{ij}.$$

This definition encapsulates the following intuition: when an auction mechanism implements a discriminatory price dominated rule, no bidder will be asked to pay more than she is currently willing to or have an incentive to do so.

There are two prevailing pricing rules in the literature of multi-unit auctions: *discriminatory pricing* and *uniform pricing*.

(a) *Discriminatory Pricing*. Every bidder $i \in \mathcal{N}$ pays for every unit won a price equal to her bid for that item, i.e. $P_i(x_i(\mathbf{b})) = \sum_{j=1}^{x_i(\mathbf{b})} = b_{ij}$. This is a generalization of the first price auction presented in section 2.3.

(b) *Uniform Pricing*. Every bidder $i \in \mathcal{N}$ pays for every unit won an identical price $p_i(\mathbf{b})$ that is the *highest losing bid* submitted to the auctioneer. Thus, $p_i(\mathbf{b}) = \max_{\ell \in \mathcal{N}} b_{i(x_\ell(\mathbf{b})+1)}$ and $P_i(\mathbf{b}) = x_i(\mathbf{b}) \cdot p_i(\mathbf{b})$. An auction type with this pricing rule can be considered a generalization of the second price auction.

Remark 2.1. *While the uniform price auction is a generalization of the Second Price Auction we have presented in 2.3, it is not a generalization of the Vickrey auction as it does not guarantee truthfulness and strategyproofness; bidders may have incentives to bid strategically. In the generalization of the Vickrey auction, each bidder is asked to pay the value of the bids that would have won if the bidder did not participate in the auction. This mechanism is truthful and is essentially an instantiation of the Vickrey-Clarke-Groves mechanism. We refer the reader to the seminal work of Vickrey [1961] for more information.*

An interesting direction of research could be to study other pricing rules than the the two rules defined above. However, it is mandatory that these new rules adhere to the definition 2.10 since, otherwise, the multi-unit auction does not longer guarantee *individual rationality*. Individual rationality is a fundamental notion of mechanism design, which in our context means that bidders cannot end-up with a negative utility as a result of rational behavior on their part.

2.4.5 *Incomplete Information*

Under the incomplete information model of Harsanyi, the valuation function $\mathbf{v}_i$ of bidder $i \in \mathcal{N}$ is drawn from a finite set V_i according to a discrete probability distribution $\pi_i : V_i \mapsto [0,1]$. This implies that bidder $i \in \mathcal{N}$ draws a probability distribution independently of other bidders. Moreover, the actual drawn function of every bidder is *private information*. Every valuation profile $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \in \mathcal{V} = \times_{i \in \mathcal{N}} V_i$ is drawn from a distribution $\pi : \mathcal{V} \mapsto [0,1]$ that is *public knowledge*. Therefore, in the multi-unit auctions model, under incomplete information each bidder i knows her own valuation function but may only estimate the valuation profiles of all other bidders by her knowledge of the publicly available distribution π. Given the publicly known distribution π, the strategy of each bidder is a function of $\mathbf{v}_i$, $B_i(\mathbf{v}_i)$. This function maps a valuation function $\mathbf{v}_i \in V_i$ to a *distribution* $B_i(\mathbf{v}_i) = B_i^{\mathbf{v_i}}$, over all possible bid vectors of i (mixed strategies included). We will write $\mathbf{b}_i \sim B_i^{\mathbf{v}_i}$.

A Bayes-Nash equilibrium in this model is a strategy profile $\mathbf{B} = (B_1, B_2, \ldots, B_n)$ such that given the public distribution of other bidder $\mathbf{w}_{-i}$ *given* her own $\mathbf{v}_i$ and over the distribution $\mathbf{B}^{(\mathbf{v_i}, \mathbf{w_{-i}})}$, for every bidder $i \in \mathcal{N}$ and for every valuation $\mathbf{v}_i$ of i, there exists a pure strategy $\mathbf{c}_i$ of i:

$$\mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i, \mathbf{b} \sim \mathbf{B}^{(\mathbf{v}_i, \mathbf{w}_{-i})}} \left[u_i^{v_i}(\mathbf{b}) \right] \geq \mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i, \mathbf{b}_{-i} \sim \mathbf{B}^{\mathbf{w}_{-i}}} \left[u_i^{v_i}(\mathbf{c}_i, \mathbf{b}_{-i}) \right],$$

where $\mathbb{E}_{\mathbf{v}}$ and $\mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i}$ are the expectations over distributions π and $\pi(\cdot|\mathbf{v}_i)$ respectively.

Fix a valuation profile $\mathbf{v} \in \mathcal{V}$ and consider a bidding configuration $\mathbf{B}^{\mathbf{v}}$. The Social Welfare is $SW(\mathbf{v}, \mathbf{B}^{v}) = \mathbb{E}_{\mathbf{b} \sim \mathbf{B}^{\mathbf{v}}}[\sum_{i \in \mathcal{N}} v_i(x_i(\mathbf{b}))]$. The expected Social Welfare in a Bayes-Nash equilibrium $\mathbf{B}^{\mathbf{v}}$ is $\mathbb{E}_{\mathbf{v} \sim \pi}[SW(\mathbf{v}, \mathbf{B}^{\mathbf{v}})]$. We denote the socially optimal allocation as $\mathbf{x}^{\mathbf{V}}$ under profile $\mathbf{v} \in \mathcal{V}$. The expected optimum is then $\mathbb{E}_{\mathbf{v} \sim \pi}[SW(\mathbf{v}, \mathbf{B}^{\mathbf{v}})]$. Hence, the *Bayesian Price of Anarchy* is

$$PoA = \sup_{\pi} \sup_{\mathbf{B}} \frac{\mathbb{E}_{\mathbf{v} \sim \pi}[SW(\mathbf{v}, \mathbf{B}^{\mathbf{v}})]}{\mathbb{E}_{\mathbf{v} \sim \pi}[SW(\mathbf{v}, \mathbf{B}^{\mathbf{v}})]}.$$

General Existence of Mixed and Bayes-Nash equilibria It is important to clarify that it cannot be assumed that mixed Nash equilibria are guaranteed to exist as a corollary of theorem 2.1[7]. Nash's theorem requires a setting where players have finite strategies. In the multi-unit auctions model, bidders have a continuous strategy space (individual bids should be non-negative). Furthermore, under the model of incomplete information the type space is also continuous since we consider continuous valuation functions.

This involved and interesting discussion is out of scope of this thesis. However, following the discussion from Feldman et al. [2013] and Hassidim et al. [2012], we overcome this

7 Of course, when pure Nash equilibria are guaranteed to exist the same holds for mixed Nash equilibria, since any pure Nash equilibrium can be regarded as a trivial mixed Nash equilibrium.

problem by using the following assumption when reasoning about classes beyond pure Nash equilibria.

Assumption 2.1. *We assume a sufficiently fine discretization of the bidding strategy space and the valuation space for all bidders. Then, strategies are finite and existence of mixed and Bayes Nash equilibria is guaranteed as a corollary of Nash's Theorem.*

With this assumption in place we transform the continuous space of bidding strategies and valuations to a discrete one. Imagine we partition the strategy space by a small $\epsilon > 0$. Then, we assume that the allowed strategies are *only* $0, \epsilon, 2\epsilon, \ldots$. Having this finite strategy space for each bidder, guarantees the existence of mixed Nash equilibria as a corollary of theorem 2.1. Moreover, the upper bounds on the price of anarchy on these classes still hold with the small difference of ϵ. In essence, the upper bounds presented for the mixed and Bayes-Nash equilibria are ϵ-approximations of the price of Anarchy, for a very small $\epsilon > 0$.

2.4.6 *Related Work*

The first time the Bayesian price of anarchy of the discriminatory price auction of was studied was by Syrgkanis and Tardos [2013], where the authors presented an upper bound on the Bayesian Price of Anarchy of $\frac{2e}{e-1}$. In this work, the authors tailored the *smoothness framework* of Roughgarden [2012]. It is noticeable that apart from de Keijzer et al. [2013]; Christodoulou et al. [2016], there are no other works regarding subadditive valuation functions.

Another auction format related to the the discriminatory price auctions and the uniform price auctions are the Generalized Second Price auctions. These types of multi-unit auctions have been studied extensively by Caragiannis et al. [2012], who managed to derive almost tight upper and lower bounds.

In general, there is a resemblance of the model of multi-unit auctions, both technically and conceptually to the model of *Simultaneous Auctions* that have been studied extensively in recent years, for instance by Christodoulou et al. [2012]; Bhawalkar and Roughgarden [2011]; Feldman et al. [2013]; Hassidim et al. [2011] and Christodoulou et al. [2016][8]. However, the upper bounds on the Price of Anarchy of these formats do not carry over to the setting of multi-unit auctions. Simultaneous auctions were studied for the first time by Christodoulou et al. [2012]. In this setting, different items are simultaneously sold in *Second Price Auctions*. The authors derive a tight upper bound on the Bayesian Price of Anarchy of the induced Bayes-Nash equilibria when bidders have *fractionally subadditive* valuation functions (a generalization of submodular valuation functions). Subsequently, Bhawalkar and Roughgarden [2011] were the first to study simultaneous Second Price auctions for the wider class of subadditive valuation functions showing an upper bound of $\mathcal{O}(\log k)$ for the Bayesian Price of Anarchy, where k is the number of distinct goods, which was improved to 4 by Feldman et al.

8 We present the multi-unit auctions results from this work. The main goal of the work is, however, simultaneous first price auctions.

[2013]. For arbitrary valuation functions, Fu et al. [2012] showed an upper bound of 2 on the Price of Anarchy of Pure Nash Equilibria.

Simultaneous First Price Auctions were first studied by Hassidim et al. [2011], who showed that pure Nash equilibria are always efficient when they exist. Regarding mixed equilibria, they proved constants inefficiency upper bounds, while for the Bayesian Price of Anarchy they proved an upper bound of $\mathcal{O}(\log k)$ for subadditive valuation functions and $\mathcal{O}(k)$ for arbitrary valuation functions, where k is the number of distinct goods. Subsequently, Syrgkanis [2012] showed that the Bayesian Price of Anarchy is at most $\frac{e}{e-1}$ for fractionally subadditive valuations. For subadditive valuation functions, Feldman et al. [2013] derived an upper bound of 4. Finally, for submodular and subadditive valuation functions, Syrgkanis and Tardos [2013] showed upper bounds of $\frac{e}{e-1}$ and 2 for the Bayesian Price of Anarchy, which was shown to be tight by Christodoulou et al. [2016].

The *smoothness framework* was developed independently by Roughgarden [2009] and Syrgkanis and Tardos [2013]. This frameworks can be applied to multi-unit auctions mechanisms and provides a template for showing Price of Anarchy results. This framework can also be used to analyze the inefficiency of simultaneous and sequential compositions of simple mechanisms. For submodular valuation functions, they showed upper bounds on compositions of the Discriminatory Price Auction and the Uniform Price Auction of $\frac{2e}{e-1}$ and $\frac{4e}{e-1}$. These results where improved by de Keijzer et al. [2013] to $\frac{e}{e-1}$ and $-\mathcal{W}_{-1}(-e^{-2}) \approx 3.146$, where $-\mathcal{W}_{-1}$ is the lower branch of the Lambert W function.

Another interesting line of work is the study of the multi-unit auction model from a mechanism design perspective. In this context, researchers seek to design truthful and computationally efficient multi-unit auctions mechanisms that output the optimal or approximately optimal allocation that maximizes the social welfare of the bidders. Prominent works are those from Dobzinski and Dughmi [2009]; Dobzinski and Nisan [2007] and Vöcking [2012].

3 EQUILIBRIA UNDER DISCRIMINATORY PRICING

OUTLINE

In this chapter we study the equilibria of multi-unit auctions with a discriminatory pricing rule, where every bidder pays the sum of her winning bids. We discuss the existence of equlibria and the price of anarchy of this auction format per class of equilibria. The results in this section are mainly due to de Keijzer et al. [2013] and Christodoulou et al. [2016].

3.1 PURE NASH EQUILIBRIA

Pure Equilibria of this auction format have been extensively analyzed in de Keijzer et al. [2013]. Since the Discriminatory Price Auction is a generalization of the First Price Auction, there is no guarantee that an arbitrary instance of this auction possesses pure Nash equilibria, in accordance to the single-item case analyzed in section 2.3.

However, for the generalized model, de Keijzer et al. [2013] show that with an appropriate tie-breaking rule there exist bidding profiles that are pure Nash equilibria.

Proposition 3.1 (de Keijzer et al. [2013])**.** *For every Discriminatory Auction there is a tie-breaking rule inducing a uniform bidding profile that is a pure Nash equilibrium under that tie-breaking rule.*

Remark 3.1. *Even though there is no guarantee that an arbitrary instance possesses a pure Nash equilibrium, de Keijzer et al. [2013] show that every discriminatory price auction possesses a pure ϵ-equilibrium. In such an equilibrium, players may have a small incentive to deviate unilaterally. However, such a deviation would yield the deviating bidder an additional utility of no more than $\epsilon > 0$, for an arbitrary value ϵ.*

When pure Nash equilibria exist in a discriminatory price auction, the utilitarian social welfare achieved at these points is always optimal. This implies that the price of anarchy of pure equilibria is 1 or, equivalently, the auction format is *efficient*, even though bidders may have incentives to bid strategically (non-truthfully). To prove this result, the authors exploit several properties of the pure equilibria of this auction format. Essentially, these properties imply that all pure equilibria in the discriminatory price auction occur under a *uniform bidding* profile. We present the properties in the following lemma.

Lemma 3.2 (de Keijzer et al. [2013])**.** *Let* $\mathbf{b}$ *be a pure Nash equilibrium in a given Discriminatory Auction where the bidders have general valuation functions. Let* $d = \max\{b_{ij} : i \in \mathcal{N}, j = 1, \dots, k, j > x_i(\mathbf{b})\}$*. Then:*

(a) *For any bidder $i \in \mathcal{N}$ who wins at least one item under* $\mathbf{b}$*, and for all* $j \in [x_i(\mathbf{b})]$ *it holds that* $b_{ij} = d$,

(b) $\ell d \leq \sum_{j=x_i(\mathbf{b})-\ell+1}^{x_i(\mathbf{b})} m_{ij}$*, for all* $i \in \mathcal{N}$ *and* $\ell \in [x_i(\mathbf{b})]$,

(c) $\sum_{j=x_i(\mathbf{b})+1}^{x_i(\mathbf{b})+\ell} m_{ij} \leq \ell d$*, for all* $i \in \mathcal{N}$ *and* $\ell \in [k - x_i(\mathbf{b})]$.

We can now state and prove the theorem about the efficiency of pure Nash equilibria under discriminatory pricing.

Theorem 3.3 (de Keijzer et al. [2013])**.** *Pure Nash equilibria of the Discriminatory Auction (with the standard or the uniform bidding interface) are always efficient, even for bidders with arbitrary valuation functions.*

Proof. Let $\mathbf{b}^*$ be an optimal bid vector in terms of social welfare. Denote by A the set of bidders that are allocated more items under $\mathbf{b}$ than under $\mathbf{b}^*$. For a bidder $i \in A$, define ℓ_i as the number of extra items that i gets under $\mathbf{b}$ when compared to $\mathbf{b}^*$; i.e., $\ell_i = x_i(\mathbf{b}) - x_i(\mathbf{b}^*)$. Denote by B the set of bidders that are allocated less items under $\mathbf{b}$ than under $\mathbf{b}^*$; i.e. $\ell_i = x_i(\mathbf{b}^*) - x_i(\mathbf{b})$. Then by summing over all bidders in $\mathcal{N}$ we have,

$$\sum_{i=1}^{n} v_i(x_i(\mathbf{b})) - \sum_{i=1}^{n} v_i(x_i(\mathbf{b}^*)) = \sum_{i=1}^{n} \left(\sum_{j=1}^{x_i(\mathbf{b})} m_{ij} - \sum_{j=1}^{x_i(\mathbf{b}^*)} m_{ij} \right)$$
$$= \sum_{i \in A} \sum_{j=x_i(\mathbf{b})-\ell_i+1}^{x_i(\mathbf{b})} m_{ij} - \sum_{i \in B} \sum_{j=x_i(\mathbf{b})+1}^{x_i(\mathbf{b})+\ell_i} m_{ij} \geq \sum_{i \in A} \ell_i d - \sum_{i \in B} \ell_i d.$$

The inequality in the derivation above follows from points (b) and (c) of Lemma 3.2. Moreover, all extra units won by bidders in A have been "stolen" from bidders in B;i.e. $\sum_{i \in A} \ell_i d = \sum_{i \in B} \ell_i d$. Therefore, the final outcome of the above derivation is that

$$\sum_{i=1}^{n} v_i(x_i(\mathbf{b})) \geq \sum_{i=1}^{n} v_i(x_i(\mathbf{b}^*)).$$

Thus, since $\mathbf{b}$ is an arbitrary pure Nash equilibrium, we conclude that the discriminatory price auction is always efficient in terms of social welfare. □

3.2 MIXED EQUILIBRIA

In the last section we have shown that whenever there are pure Nash equilibria, they are efficient. In this section we will show that for the class of mixed Nash Equilibria that is no longer the case. We present two explicit constructions of instances that are inefficient mixed equilibria by Christodoulou et al. [2016]. Upper bounds on the price of anarchy of mixed equilibria can be inferred by upper bounding the Bayesian price of anarchy. We will present these results in the section that follows.

3.2.1 *Submodular Valuation Functions*

We start by presenting lower bounds on the Price of Anarchy of mixed equilibria. Christodoulou et al. [2016] show that the Price of Anarchy of mixed Equilibria for submodular bidders is at least 1.099 (no longer 1).

Theorem 3.4 (Christodoulou et al. [2016])**.** *The price of anarchy of mixed Equilibria of the discriminatory price auction when bidders have submodular valuation functions is at least 1.099.*

Proof. Let $v \in (1/2, 2]$. Let $z \in [0, 1/2]$ be a random variable following the cumulative distribution function (CDF)

$$G(z) = \frac{z}{1-z}, \quad z \in [0, 1/2].$$

Let $y \in [0, 1/2]$ be a random variable following the cumulative distribution function (CDF)

$$F(y) = \frac{v - 1/2}{v - y}, \quad y \in [0, 1/2].$$

We design a game with two bidders and two units.

$$\begin{aligned} \mathbf{m}_1 &= (v, 0), \quad \mathbf{b}_1 = (z, 0) \\ \mathbf{m}_2 &= (1, 1), \quad \mathbf{b}_2 = (y, y) \end{aligned}$$

Note that bidder 2 bids 0 with probability $F(0) = 1 - 1/2v$. We set the following tie-breaking rule: when bidder 2 bids 0 she gets 1 item. We claim that this bidding profile of mixed strategies is a mixed Nash equilibrium by showing that bidding vectors $(z, 0)$ and (y, y) are the best responses of each bidder to the bids of others.

The expected utility of bidder 1 is:

$$\mathbb{E}[u_1(z, 0)] = \Pr[z > y](v - z) = F(z)(v - z) = v - 1/2.$$

For bidder 1, bidding (z, z') with $z' \leq z$ would only introduce a price for the second unit; the bidder is indifferent towards acquiring an additional unit. Moreover bidding $(z', 0)$ with $z' > 1/2$ would guarantee her the item but at the same time introduce a high payment.

Now we need to show that (y, y) is a best response for bidder 2. The expected utility of bidder 2 at (y, y) is

$$\mathbb{E}[u_2(y, y)] = G(y)(1 - y) = 1.$$

Consider any strategy (y, y') with $y, y' \in [0, 1/2]$ and $y \geq y'$.

$$\begin{aligned}
\mathbb{E}[u_2(y, y')] &= \Pr[z \leq y'](2 - y - y') + \Pr[x > y'] \\
&= G(y')(2 - y - y') + (1 - G(y'))(1 - y) \\
&= 1 + y' - y \leq 1 = \mathbb{E}[u_2(y, y)].
\end{aligned}$$

Bidding strictly higher than $1/2$ for both items is not profitable since it would yield a utility $2 - 2y < 1$. Therefore, since both bidders have no incentive to deviate unilaterally, we conclude that this bidding profile is a mixed Nash equilibrium. □

The optimal social welfare of this auction instance is clearly 2 (bidder 2 acquires both items). The expected social welfare of this bidding profile is

$$\begin{aligned}
\mathbb{E}[SW] &= \Pr[y \geq z]2 + \Pr[y < z](1 + v) \\
&= 2 - (1 - v)\Pr[y < z] \\
&= 2 - (1 - v)\int_0^{1/2} F(z)dG(z).
\end{aligned}$$

This expression is minimized for $v = 0.643$, for which $\mathbb{E}[SW] = 1.818$. There the price of anarchy of the discriminatory price auction is at least $2/1.818 = 1.099$.

3.2.2 *Subadditive Valuation Functions*

For bidders with subadditive valuation functions, the authors have presented a different construction that achieves roughly half of the optimal social welfare.

Theorem 3.5 (Christodoulou et al. [2016]). *The price of anarchy of mixed Equilibria of the discriminatory price auction when bidders have subadditive valuation functions is at least*

$$PoA \geq \frac{2}{1 + \frac{2}{\sqrt{k}} - \frac{1}{k}}$$

and approaches 2 as k grows.

Proof Sketch. Let $v \in (0, 1]$. Let $z \in [0, 1/k]$ be a random variable following the cumulative distribution function (CDF)

$$G(z) = \frac{(k-1)z}{1-z}, \quad z \in [0, 1/k].$$

Let $y \in [0, 1/k]$ be a random variable following the cumulative distribution function (CDF)

$$F(y) = \frac{v - 1/k}{v - y}, \quad y \in [0, 1/k].$$

We design a game with two bidders and k units.

$$\mathbf{m}_1 = (v, 0, \ldots, 0, 0), \quad \mathbf{b}_1 = (z, 0, \ldots, 0)$$
$$\mathbf{m}_2 = (1, 0, \ldots, 0, 1), \quad \mathbf{b}_2 = (y, y, \ldots, y)$$

Note that bidder 2 bids 0 with probability $F(0) = 1 - 1/kv$. We set the following tie-breaking rule: when bidder 2 bids 0 she gets 1 item.

In a similar approach to the one described in theorem 3.5, authors prove that this instance is a mixed Nash Equilibrium that achieves a social welfare $\frac{1}{2}$ of the optimal solution as k grows. □

Upper bounds are also known for mixed Nash equilibria, both for submodular and subadditive valuation functions. We present them in the next subsection where we reason about Price of Anarchy bounds for Bayes-Nash equilibria, a class that contains mixed equilibria.

3.3 BAYES-NASH EQUILIBRIA

In this section we present upper bounds on the Bayesian Price of Anarchy of the discriminatory price auction. These upper bounds were derived initially by de Keijzer et al. [2013] and are presented in the following theorem.

Theorem 3.6 (de Keijzer et al. [2013])**.** *The Bayesian Price of Anarchy of the Discriminatory Price Auction (under the standard or uniform bidding format) is at most* $\frac{e}{e-1} \approx 1.582$ *and* $\frac{2e}{e-1} = 3.164$ *for submodular and subadditive valuation functions, respectively.*

3.3.1 *A Framework for Bayesian Price of Anarchy Upper Bounds*

To derive upper bounds on the Bayesian Price of Anarchy, de Keijzer et al. [2013] presented a unified treatment of both the discriminatory price auction and the uniform price auction and both submodular and subadditive bidders. In particular, they proposed a framework for deriving upper bounds for the Bayesian Price of Anarchy of both auctions in the form of the following theorem.

Theorem 3.7 (de Keijzer et al. [2013])**.** *Let V be a class of valuation functions. Suppose that every valuation profile $\mathbf{v} \in V^n$, for every bidder $i \in \mathcal{N}$, and for every distribution $\mathcal{P}_{-i}$ over* *non-overbidding profiles* $\mathbf{b}_{-i}$*, there is a bidding profile* $\mathbf{b'}_i$ *such that the following inequality holds for some* $\lambda > 0$ *and* $\mu \geq 0$*:*

$$\mathbb{E}_{\mathbf{b}_{-i} \sim \mathcal{P}_{-i}} \left[u_i^{\mathbf{v}_i}(\mathbf{b'}_i, \mathbf{b}_{-i}) \right] \geq \lambda \cdot v_i(x_i^{\mathbf{v}}) - \mu \cdot \mathbb{E}_{\mathbf{b}_{-i} \sim \mathcal{P}_{-i}} \left[\sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}) \right]. \tag{3.1}$$

Then, the Bayesian Price of Anarchy is at most

(a) $\max\{1, \mu\}/\lambda$ *for the Discriminatory Price Auction.*

(b) $(\mu + 1)/\lambda$ *for the Uniform Price Auction.*

This theorem implies than when searching for an upper bound on the Bayesian Price of Anarchy, one should concentrate on finding a unilateral deviation of any bidder i with the property of equation (3.1). Note that the framework of Theorem 3.7 holds for both bidding interfaces (standard and uniform bidding) as there is constraint on the chosen deviating vector $\mathbf{b'}_i$. Moreover, there are no assumptions on the nature of bidders valuation functions (submodular or subadditive).

Additionally, de Keijzer et al. [2013] provide a specific instantiation of their framework by determining an appropriate bid vector $\mathbf{b'}_i$ for every bidder $i \in \mathcal{N}$.

Lemma 3.8 (de Keijzer et al. [2013])**.** *Let* v *be a valuation profile and suppose that the pricing rule is discriminatory price dominated. Define* $\tau_i = \arg\min_{j \in [x_i^{\mathbf{v}}]} v_i(j)/j$ *for every* $i \in \mathcal{N}$*. Then, for every bidder* $i \in \mathcal{N}$ *and every bidding profile* $\mathbf{b}_{-i}$*, there exists a randomized uniform bidding profile strategy* B'_i*, such that for every* $\alpha > 0$

$$\mathbb{E}_{\mathbf{b'}_i \sim B'_i} \left[u_i^{\mathbf{v}_i}(\mathbf{b'}_i, \mathbf{b}_{-i}) \right] \geq \alpha \left(1 - \frac{1}{e^{\frac{1}{\alpha}}}\right) x_i^{\mathbf{v}} \frac{v_i(\tau_i)}{\tau_i} - \alpha \sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}). \tag{3.2}$$

The deviation B'_i used in Lemma 3.8 is a mixed strategy. Every mixed strategy can be regarded as a distribution of pure strategies. Since the lemma holds for *every* pure bidding profile $\mathbf{b}_{-i}$, we can take expectation on both sides of inequality (3.2) over any distribution $\mathcal{P}_{-i}$ of such profiles $\mathbf{b}_{-i}$. Then, we arrive to a version of inequality (3.1) that is required by theorem 3.7. This means that there is at least one pure strategy $\mathbf{b'}_i$ in B'_i satisfying (3.1) for

$$(\lambda, \mu) = \left(\alpha \left(1 - \frac{1}{e^{\frac{1}{\alpha}}}\right), \alpha\right).$$

Proof of Theorem 3.4. For submodular valuation we notice that we obtain the indicated upper bound by setting $\alpha = 1$. To obtain an upper bound on subadditive valuation functions, de Keijzer et al. [2013] show that

$$\frac{v_i(\tau_i)}{\tau_i} \geq \frac{1}{2} \frac{v_i(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} \tag{3.3}$$

by using the definition 2.9. Therefore, for subaddditive valuation functions, theorem 3.7 holds for

$$(\lambda, \mu) = \left((\frac{\alpha}{2} \left(1 - \frac{1}{e^{\frac{1}{\alpha}}} \right), \alpha \right)$$

and, thus, by setting once again $\alpha = 1$ the theorem follows. □

The Smoothness Framework The method for deriving upper bounds proposed by theorem 3.7 falls into the definition of the Smoothness Framework developed by Roughgarden [2009] for general games and tailored to mechanisms by Syrgkanis and Tardos [2013].

Definition 3.1 (Syrgkanis and Tardos [2013]). *A mechanism $\mathcal{M}$ is (λ, μ)-smooth for $\lambda > 0$ and $\mu \geq 0$ if for any valuation profile $\mathbf{v}$ and for any bidding profile $\mathbf{b}$ there exists a randomized bidding strategy $B'_i(\mathbf{v}, \mathbf{b}_i)$ for each bidder i, such that*

$$\sum_{i \in \mathcal{N}} \mathbb{E}_{\mathbf{b}'_i \sim B'_i} \left[u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i}) \right] \geq \lambda \cdot SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \cdot \sum_{i \in \mathcal{N}} P_i(\mathbf{b}). \tag{3.4}$$

Using lemma 3.8, de Keijzer et al. [2013] show that theorem 3.7 is an instantiation of this framework. Moreover, Syrgkanis and Tardos [2013] provide identical expressions in terms of (λ, μ) for upper bounding the Bayesian Price of Anarchy with the expression in theorem 3.7 for the discriminatory price auction. Additionally, there are some interesting automatic results regarding (λ, μ)-smooth mechanisms and their sequential and simultaneous compositions. This involved topic is, however, out of the scope of this thesis.

3.3.2 *Subadditive Valuation Functions Under Standard Bidding*

The results of the previous section hold for both the standard and the uniform bidding interface. The upper bound of 3.164 can be improved to 2. The approach of de Keijzer et al. [2013] is inspired by the one in Feldman et al. [2013] for simultaneous auctions. The idea is to construct a bid $\mathbf{b}'_i$ by using the distribution $\mathcal{P}_{-i}$ of profiles $\mathbf{b}_{-i}$.

Lemma 3.9 (de Keijzer et al. [2013]). *Let V be the class of subadditive valuation functions. The Theorem 3 holds true with $(\lambda, \mu) = (\frac{1}{2}, 1)$ for the Discriminatory Price Auction.*

4

EQUILIBRIA UNDER UNIFORM PRICING

OUTLINE

In this chapter we study the equilibria of multi-unit auctions with a uniform pricing rule, where every bidder pays the highest losing bid that was submitted to the auctioneer. We discuss the existence of equlibria and the price of anarchy of this auction format per class of equilibria. Under this format, only a subset of pure equilibria called undominated equilibria were studied before by Markakis and Telelis [2015]. We develop tight welfare guarantees for the general class of pure equilibria under the standard bidding format, by deriving an upper bound on the Price of Anarchy along with an explicit construction that matches the upper bound. These results are the novel contributions of this thesis and are included in Birmpas et al. [2017] (joint work of the author along with Georgios Birmpas, Evangelos Markakis and Orestis Telelis). Additionally, we provide an alternative derivation to the one presented in de Keijzer et al. [2013] that adheres to the framework of theorem 3.7 and matches the currently-known upper bound of 3.146. This derivation is inspired by a similar derivation for the discriminatory price auction that was carried out by Christodoulou et al. [2016]. To the best of our knowledge, this is the first time this derivation is presented. We conclude the section with results regarding subadditive valuation functions from de Keijzer et al. [2013].

4.1 PURE EQUILIBRIA

Recall that in the scope of this thesis we study equilibria of strategies where bidders do not overbid. In particular, recall that we consider bidding vectors $\mathbf{b}$ wherein no bidder ever outbids her value, for any number of units. That is, for any $\ell \leq k$, we assume

$$\sum_{j=1}^{\ell} b_{ij} \leq v_i(\ell).$$

When studying the equilibria of the uniform price auction, this is particularly important since there can be equilibria when bidders violate this assumption (contrary to the discriminatory price auction where overbidding is an irrational behavior). In fact, this may lead to arbitrary bad equilibria in terms of social welfare, making the price of anarchy *unbounded*. It can be assumed that bidders are risk-averse by nature and are aware that overbidding may lead to dominated equilibria.

4.1.1 *Existence of Pure Nash Equilibria*

In contrast to the Discriminatory Price Auction, for bidders with submodular valuation functions, it is guaranteed that there always exists an efficient pure Nash equilibrium for every instance of the game. This can be inferred by the fact that this auction format is a generalization of the second price auction for a single item, where bidding truthfully admits a pure Nash equilibrium. We will now prove this claim for the uniform price auction.

Claim 4.1. *Let* $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ *be an optimal allocation for any instance of the Uniform Price auction with* $k > 1$ *items and* n *bidders in which bidders have submodular valuation functions. Then, the profile* $\mathbf{b}$ *with* $\mathbf{b}_i = (m_{i1}, \ldots, m_i(x_i^*), 0, \ldots, 0)$ *if* $x_i^* \geq 1$ *and* $\mathbf{b}_i = \mathbf{0}$ *otherwise, is an efficient pure Nash Equilibrium.*

Proof. When bidders bid $\mathbf{b}$, notice that the uniform price (highest losing bid) is 0 and the allocation $\mathbf{x}^*$.

Fix a bidder $i \in \mathcal{N}$. Notice that i has no incentive to submit a different bid since she is indifferent towards the rest $k - x_i^*$ items and bidding for less items could only result to the loss of those items. Note that i cannot hope to get a price reduction since the price is already 0. Therefore bidders have no unilateral strategy b_i' that improves their utility. The social welfare of $\mathbf{b}$ is optimal since $\mathbf{x}(\mathbf{b}) = \mathbf{x}^*$. □

This claim does not hold for subadditive valuation functions and it remains unknown whether every instance of the Uniform Price Auction has a pure Nash equilibrium when bidders have subadditive valuation functions.

4.1.2 *Undominated Pure Nash Equilibria*

Markakis and Telelis [2015] studied a subclass of pure Nash equilibria called *undominated equilibria* for submodular valuation functions. The characterization of this subclass of equilibria begins from the two lemmas that follow.

Lemma 4.1 (Markakis and Telelis [2015])**.** *For bidders with submodular valuation functions, and for any* $i \in [k]$*, it is a weakly dominated strategy to declare a bid* b_{ij} *such that* $b_{ij} > m_{ij}$*.*

Note that such bids do not necessarily violate inequality (2.2). One could say they represent a *conservative behavior* on the part of the bidders.

Lemma 4.2 (Markakis and Telelis [2015])**.** *In an undominated strategy, a bidder with a submodular valuation never declares a bid* $b_{i1} \neq m_{i1}$*.*

Then, Markakis and Telelis [2015] provide a characterization of the pure Nash equilibria that occur in this setting. This characterization leads them to provide tight welfare guarantees for these types of equilibria. We present their result in the following theorem.

Theorem 4.3 (Markakis and Telelis [2015]). *The Price of Anarchy of undominated pure Nash equilibria of the Uniform Price Auction with submodular bidders is almost* $\frac{e}{e-1} \approx 1.582$.

In the subsections that follow, we drop the assumption that bidders submit only undominated strategy. Our only working assumption is the no-overbidding assumption of inequality (2.2). Hence, from now on we capture all feasible pure Nash Equilibria.

4.1.3 *Inefficiency Upper Bound for Submodular Bidders*

In this subsection we develop tight welfare guarantees for feasible (non-overbidding) pure Nash equilibrium profiles of the uniform price auction, when the bidders have submodular valuation functions. By the results of de Keijzer et al. [2013], it is already known for submodular valuation functions on k units that $2 - \frac{1}{k} \leq PoA \leq 3.146$. We show that:

Theorem 4.4. *The Price of Anarchy of non-overbidding pure Nash equilibria of the Uniform Price Auction with submodular bidders is at most:*

$$\frac{2 + \mathcal{W}_0(-e^{-2})}{1 + \mathcal{W}_0(-e^{-2})} \approx 2.188$$

where $\mathcal{W}_0$ *is the first branch of the Lambert W function.*

The Lambert W function is the multi-valued inverse function of $f(W) = We^W$, Corless et al. [1996]. We introduce first some notation to be used throughout the section. Let $\mathbf{b}$ denote a feasible bidding profile. We denote the winning (marginal) bids under $\mathbf{b}$ by $\beta_j(\mathbf{b})$, $j = 1, \ldots, k$, so that $\beta_j(\mathbf{b})$ is the j-th lowest winning bid under $\mathbf{b}$, thus, $\beta_1(\mathbf{b}) \leq \beta_2(\mathbf{b}) \leq \cdots \leq \beta_k(\mathbf{b})$. We will often apply this notation to profiles of the form $\mathbf{b}_{-i}$, for some bidder $i \in \mathcal{N}$. For a profile of valuation functions $(v_1, v_2, \ldots, v_n)$, we denote the *socially optimal* – i.e., welfare maximizing – allocation by $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$. If there are multiple such allocations, we fix one for the remainder of the analysis. We define a partition of the set of bidders, $\mathcal{N}$, with reference to $\mathbf{x}^*$ and any arbitrary allocation $\mathbf{x}$, into two subsets, $\mathcal{O}$ and $\mathcal{U}$, as follows:

$$\mathcal{N} = \mathcal{O} \cup \mathcal{U}, \quad \mathcal{O} = \{i \in \mathcal{N} : x_i \geq x_i^*\}, \quad \mathcal{U} = \{i \in \mathcal{N} : x_i < x_i^*\}.$$

The set $\mathcal{O}$ contains the *"overwinners"*, i.e., bidders that receive in $\mathbf{x}$ at least as many units as in $\mathbf{x}^*$. The set $\mathcal{U}$ contains respectively the *"underwinners"*. In our analysis, the allocations we refer to are determined by some profile $\mathbf{b}$, i.e., $\mathbf{x} \equiv \mathbf{x}(\mathbf{b})$. Consequently, the sets $\mathcal{O}$ and $\mathcal{U}$ will depend on $\mathbf{b}$; for simplicity, we omit this dependence from our notation. The following lemma states that, under a feasible bidding profile $\mathbf{b}$, every bidder $i \in \mathcal{O}$ retains value at least equal to a convex combination of her socially optimal value, $v_i(x_i^*)$, and of the sum of her winning bids.

Lemma 4.5. *Let* $\mathbf{b}$ *be a feasible bidding profile, and let* $\mathcal{O}$ *be the set of overwinners with respect to the allocation* $\mathbf{x}(\mathbf{b})$*. Then, for every* $\lambda \in [0,1]$*, and for every bidder* $i \in \mathcal{O}$*, we have:*

$$v_i(x_i(\mathbf{b})) \geq \lambda \cdot v_i(x_i^*) + (1-\lambda) \cdot \sum_{j=1}^{x_i(\mathbf{b})} b_{ij} \tag{4.1}$$

Proof. Indeed:

$$\begin{aligned} v_i(x_i(\mathbf{b})) &= \lambda v_i(x_i(\mathbf{b})) + (1-\lambda) v_i(x_i(\mathbf{b})) \\ &\geq \lambda v_i(x_i^*) + (1-\lambda) v_i(x_i(\mathbf{b})) \end{aligned}$$

by definition of $\mathcal{O}$. Then, (4.1) follows by our no-overbidding assumption on $\mathbf{b}$.

□

By definition, each overwinner is capable of *"covering"* her socially optimal value. Conversely, the underwinners are the cause of social inefficiency. We will bound the total inefficiency by transforming the leftover fractions of winning bids of bidders in $\mathcal{O}$, i.e., the term $(1-\lambda) \cdot \sum_{j=1}^{x_i(\mathbf{b})} b_{ij}$ for each bidder $i \in \mathcal{O}$ in (4.1), into fractions of the value attained by bidders in $\mathcal{U}$ in the optimal allocation. In this manner, we will quantify the value that the underwinners are missing (due to their strategic bidding), and determine the worst-case scenario that can arise at a pure Nash equilibrium. The following claim can be inferred from Markakis and Telelis [2015], and will be used to facilitate this transformation. We present the proof for completeness.

Claim 4.1. *Let* $\mathbf{b}$ *be any bidding profile. Then it holds that:*

$$\sum_{i \in \mathcal{U}} \sum_{j=1}^{x_i^* - x_i(\mathbf{b})} \beta_j(\mathbf{b}) \leq \sum_{i \in \mathcal{O}} \sum_{j=x_i^*+1}^{x_i(\mathbf{b})} b_{ij}. \tag{4.2}$$

Proof. For every unit missed under $\mathbf{b}$ by any bidder $i \in \mathcal{U}$ (with respect to the units won by i in the optimal allocation), there must exist some bidder $\ell \in \mathcal{O}$ that obtains this unit. If i missed $x_i^* - x_i(\mathbf{b}) > 0$ units under $\mathbf{b}$, there are at least as many bids issued by bidders in $\mathcal{O}$ who obtained collectively these units. The sum of these bids cannot be less than the sum $\sum_{j=1}^{x_i^*-x_i(\mathbf{b})} \beta_j(\mathbf{b})$ of the $x_i^* - x_i(\mathbf{b})$ lowest winning bids in $\mathbf{b}$. Hence, summing over every $i \in \mathcal{U}$ yields the desired inequality. □

Next, we develop a characterization of upper bounds on the Price of Anarchy. To this end, let us first define the following set, $\Lambda(\mathbf{b})$, for any bidding profile $\mathbf{b}$.

$$\Lambda(\mathbf{b}) = \left\{ \lambda \in [0,1] : v_i(x_i(\mathbf{b})) + (1-\lambda) \sum_{j=1}^{x_i^* - x_i(\mathbf{b})} \beta_j(\mathbf{b}) \geq \lambda v_i(x_i^*), \forall i \in \mathcal{U} \right\} \tag{4.3}$$

Notice that, for every $\mathbf{b}$, $\Lambda(\mathbf{b}) \neq \emptyset$, because $\lambda = 0 \in \Lambda(\mathbf{b})$. The following simple lemma helps us understand how one can obtain upper bounds on the Price of Anarchy.

Lemma 4.6. *If there exists* $\lambda \in [0,1]$ *such that* $\lambda \in \Lambda(\mathbf{b})$, *for every feasible pure Nash equilibrium profile* $\mathbf{b}$ *of the Uniform Price Auction, then the Price of Anarchy of feasible pure Nash equilibria is at most* λ^{-1}.

Proof. Fix a feasible pure Nash equilibrium profile $\mathbf{b}$ and consider any $\lambda \in \Lambda(\mathbf{b})$. Then, we can apply consecutively the partition $\mathcal{N} = \mathcal{O} \cup \mathcal{U}$ with respect to $\mathbf{b}$, Lemma 4.5, Claim 4.1, and finally, the definition of $\Lambda(\mathbf{b})$, to obtain:

$$\begin{aligned} SW(\mathbf{b}) &= \sum_{i \in \mathcal{O}} v_i(x_i(\mathbf{b})) + \sum_{i \in \mathcal{U}} v_i(x_i(\mathbf{b})) \\ &\geq \lambda \sum_{i \in \mathcal{O}} v_i(x_i^*) + (1-\lambda) \sum_{i \in \mathcal{O}} \sum_{j=x_i^*+1}^{x_i(\mathbf{b})} b_{ij} + \sum_{i \in \mathcal{U}} v_i(x_i(\mathbf{b})) \\ &\geq \lambda \sum_{i \in \mathcal{O}} v_i(x_i^*) + \sum_{i \in \mathcal{U}} \Big((1-\lambda) \sum_{j=1}^{x_i^* - x_i(\mathbf{b})} \beta_j(\mathbf{b}) + v_i(x_i(\mathbf{b})) \Big) \\ &\geq \lambda \sum_{i \in \mathcal{O}} v_i(x_i^*) + \sum_{i \in \mathcal{U}} \lambda \cdot v_i(x_i^*) = \lambda \cdot SW(\mathbf{x}^*) \end{aligned}$$

□

Using $\lambda = 0$ with Lemma 4.6, yields the trivial upper bound of ∞. To obtain better upper bounds, Lemma 4.6 shows that we need to understand better the sets $\Lambda(\mathbf{b})$, and whether underwinners can extract at equilibrium a good fraction of their value under the optimal assignment. By the definition of these sets, the next step towards this is to derive lower bounds on every $\beta_\ell(\mathbf{b})$ for each underwinner $i \in \mathcal{U}$, and every value $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$. The lower bound that we will use is formally expressed below.

Lemma 4.7. *Let* $\mathbf{b}$ *be a pure Nash equilibrium of the Uniform Price Auction and* $\mathbf{x}^*$ *be a socially optimal allocation. For every underwinning bidder* $i \in \mathcal{U}$ *under* $\mathbf{b}$ *and for every* $\ell = 1, \cdots, x_i^* - x_i(\mathbf{b})$*:*

$$\beta_\ell(\mathbf{b}) \geq \frac{1}{x_i(\mathbf{b}) + \ell} \cdot \Big(v_i(x_i(\mathbf{b}) + \ell) - v_i(x_i(\mathbf{b})) \Big) \tag{4.4}$$

We defer the proof of this statement, in order to explain first how – along with Lemma 4.6 – it leads to the proof of Theorem 4.4.

Proof. (***of Theorem 4.4***) Using Lemma 4.6, we identify values of λ that belong to every $\Lambda(\mathbf{b})$. Fix any feasible pure Nash equilibrium profile $\mathbf{b}$ and, for every bidder $i \in \mathcal{U}$, let $q_i(\mathbf{b}) = x_i^* - x_i(\mathbf{b})$. To simplify the notation, we use hereafter x_i for $x_i(\mathbf{b})$, p for $p(\mathbf{b})$, q_i for $q_i(\mathbf{b})$, and β_j for $\beta_j(\mathbf{b})$, (always with respect to the Nash equilibrium $\mathbf{b}$).

Consider an arbitrary $\lambda \in [0,1]$ and, keeping everything else fixed, define $h(\lambda) = v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \beta_j$. We can now have the following implications.

$$\begin{aligned} h(\lambda) &= v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \beta_j \\ &\geq v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \frac{1}{j+x_i} \cdot \Big(v_i(x_i+j) - v_i(x_i) \Big) \qquad (4.5) \\ &= v_i(x_i) + (1-\lambda) \cdot \sum_{j=1}^{q_i} \left(\frac{j}{j+x_i} \cdot \frac{v_i(x_i+j) - v_i(x_i)}{j} \right) \\ &\geq v_i(x_i) + (1-\lambda) \cdot \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \cdot \sum_{j=1}^{q_i} \frac{j}{j+x_i}. \qquad (4.6) \end{aligned}$$

In the derivation above, inequality (4.5) follows by applying (4.4) from Lemma 4.7, for every β_j, $j = 1, \ldots, q_i$. Inequality (4.6) follows by application of the second statement of Proposition 2.2, which yields $\frac{v_i(x_i+j)-v_i(x_i)}{j} \geq \frac{v_i(x_i^*)-v_i(x_i)}{x_i^*-x_i}$, for any $j = 1, \ldots, q_i$.

Suppose now that under the equilibrium $\mathbf{b}$, there exists $i \in \mathcal{U}$ such that $x_i = 0$. In order for some λ to belong to $\Lambda(\mathbf{b})$, we would need to have $h(\lambda) \geq \lambda v_i(x_i^*)$. Using (4.6), for the underwinners with $x_i = 0$, and substituting $v_i(x_i) = 0$, we obtain: $h(\lambda) \geq (1-\lambda) v_i(x_i^*)$. If we now impose that $(1-\lambda) v_i(x_i^*) \geq \lambda v_i(x_i^*)$, we obtain $\lambda \leq 1/2$. Thus, any value of λ in $[0, 1/2]$ satisfies the constraint in the definition of $\Lambda(\mathbf{b})$ for bidders in $\mathcal{U}$ with $x_i = 0$. It remains to consider the more interesting case, which is for bidders in $\mathcal{U}$ with $x_i > 0$. We continue from ((4.6)) to bound $h(\lambda)$ as follows:

$$\begin{aligned} h(\lambda) &\geq \lambda v_i(x_i) + (1-\lambda) \cdot \left(v_i(x_i) + \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \cdot \sum_{j=1}^{q_i} \frac{j}{j+x_i} \right) \\ &\geq \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \frac{x_i}{x_i^* - x_i} \cdot \left(1 - \sum_{j=1}^{q_i} \frac{1}{j+x_i} \right) \right) \\ &\geq \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \frac{x_i}{x_i^* - x_i} \cdot \left(1 - \int_{x_i}^{x_i^*} \frac{1}{y} dy \right) \right) \\ &\geq \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \frac{x_i}{x_i^* - x_i} \cdot \left(1 + \ln \frac{x_i}{x_i^*} \right) \right) \\ &= \lambda \cdot v_i(x_i) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \frac{\frac{x_i}{x_i^*}}{1 - \frac{x_i}{x_i^*}} \cdot \left(1 + \ln \frac{x_i}{x_i^*} \right) \right) \end{aligned}$$

The second inequality follows from the fact that $v_i(x_i(\mathbf{b})) \geq \frac{x_i}{x_i^*-x_i} \cdot \sum_{j=x_i+1}^{x_i^*} m_{ij}$, which is an implication of the first statement of Proposition 2.2. We have bounded the sum of harmonic terms by using $\sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$, which holds for any monotonically decreasing positive function.

To continue, we minimize the function $f(y) = 1 + \frac{y}{1-y} \cdot (1 + \ln y)$ over $(0,1)$, since x_i/x_i^* belongs to this interval.

Fact 4.1. *The minimum of the function $f(y) = 1 + \frac{y}{1-y} \cdot (1 + \ln y)$ over $(0,1)$, is achieved at $y = -\mathcal{W}_0(-e^{-2})$, where $\mathcal{W}_0$ is the first branch of the Lambert W function.*

By substituting, we obtain a new lower bound on $h(\lambda)$ as follows:

$$h(\lambda) \geq \lambda \cdot v_i(x_i(\mathbf{b})) + (1-\lambda) \cdot \left(\sum_{j=x_i+1}^{x_i^*} m_{ij} \right) \cdot \left(1 + \mathcal{W}_0(-e^{-2})\right)$$

If we now set the right hand side of the above to be greater than or equal to $\lambda v_i(x_i^*)$, we can check which values of λ can belong to $\Lambda(\mathbf{b})$. In particular, we notice that by using $\lambda^* = (1 + \mathcal{W}_0(-e^{-2}))/(2 + \mathcal{W}_0(-e^{-2})) \approx 0.457$, we have that $h(\lambda^*) \geq \lambda^* v_i(x_i^*)$ for every bidder $i \in \mathcal{U}$ with $x_i > 0$. Since for bidders with $x_i = 0$, we found earlier that $\lambda \leq 1/2$ suffices, and since $\lambda^* < 1/2$, we conclude that $\lambda^* \in \Lambda(\mathbf{b})$. Hence, the theorem follows by Lemma 4.6. $\square$

To complete our analysis, we provide the proof of Lemma 4.7.

Proof. (***of Lemma 4.7***) Let $\mathbf{b}$ denote a feasible pure Nash equilibrium profile and $p(\mathbf{b})$ be the uniform price under $\mathbf{b}$. Fix an underwinning bidder $i \in \mathcal{U}$. We explore whether i is able to deviate from $\mathbf{b}$ feasibly and unilaterally to obtain ℓ additional units for $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$. Consider the following deviation $\mathbf{b}_i'$, for bidder i.

$$\mathbf{b}_i' = \Big(\underbrace{b_{i1}, \cdots, b_{ir}}_{r \text{ bids}}, \underbrace{\beta_\ell(\mathbf{b}_{-i}) + \epsilon, \beta_\ell(\mathbf{b}_{-i}) + \epsilon, \ldots, \beta_\ell(\mathbf{b}_{-i}) + \epsilon}_{x_i(\mathbf{b}) + \ell - r \text{ bids}}, 0, 0, \ldots, 0\Big),$$

where $0 \leq r \leq x_i(\mathbf{b})$ is the index of the last bid in $\mathbf{b}_i$, up to position $x_i(\mathbf{b})$, that is strictly larger than $\beta_\ell(\mathbf{b}_{-i}) + \epsilon$, and $\epsilon > 0$ is an *arbitrarily small* positive constant. The last index of $\mathbf{b}_i'$ with a value of $\beta_\ell(\mathbf{b}_{-i}) + \epsilon$ is the $(x_i + \ell)$-th bid. All subsequent bids are set to 0. Observe that such a bidding vector (should it be feasible) would grant bidder i exactly $x_i(\mathbf{b}) + \ell$ units in total in the profile $(\mathbf{b}_i', \mathbf{b}_{-i})$: the first r bids of $\mathbf{b}_i'$ were already winning bids in $\mathbf{b}$ and each of the next $x_i(\mathbf{b}) + \ell - r$ bids exceed the ℓ-th lowest winning bid of the other bidders, $\beta_\ell(\mathbf{b}_{-i})$. Moreover, the price at $(\mathbf{b}_i', \mathbf{b}_{-i})$ would be $\beta_\ell(\mathbf{b}_{-i})$; this is now the highest losing bid (issued by some other bidder in the auction).

Note that $\mathbf{b}_i'$ may not always be a feasible deviation, since it may not obey the no-overbidding assumption. We continue by examining the two cases separately.

Case 1: The bidding vector $\mathbf{b}'_i$ is a feasible deviation. Then bidder i obtains ℓ additional units by deviating. But since $\mathbf{b}$ is a pure Nash equilibrium, the utility of the bidder at $(\mathbf{b}'_i, \mathbf{b}_{-i})$ cannot be higher than the utility obtained by the bidder at $\mathbf{b}$, i.e.:

$$v_i(x_i(\mathbf{b}) + \ell) - (x_i(\mathbf{b}) + \ell) \cdot \beta_\ell(\mathbf{b}_{-i}) \leq v_i(x_i(\mathbf{b})) - x_i(\mathbf{b}) \cdot p(\mathbf{b})$$

Thus, for a bidder that may feasibly perform such a deviation, a lower bound for β_ℓ is, for $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$:

$$\beta_\ell(\mathbf{b}_{-i}) \geq \frac{1}{\ell + x_i(\mathbf{b})} \cdot \Big(v_i(x_i(\mathbf{b}) + \ell) - v_i(x_i(\mathbf{b})) + x_i(\mathbf{b}) \cdot p(\mathbf{b})\Big)$$

By dropping the non-negative term $x_i(\mathbf{b}) \cdot p(\mathbf{b})$ and since $\beta_\ell(\mathbf{b}) \geq \beta_\ell(\mathbf{b}_{-i})$ for every $\ell = 1, \ldots, x_i^* - x_i(\mathbf{b})$, we obtain (4.4).

Before continuing to examine the second case, concerning a *non-feasible* $\mathbf{b}'_i$, we identify first a useful inequality pertaining exclusively to feasible bidding vectors:

Claim 4.1. *The vector $\mathbf{b}'_i$ satisfies the no-overbidding assumption if and only if*

$$v_i(x_i(\mathbf{b}) + \ell) \geq \sum_{j=1}^{x_i(\mathbf{b})+\ell} b'_{ij}$$

Proof. If $\mathbf{b}'_i$ is a feasible deviation, the inequality holds, by definition of no-overbidding. For the reverse direction, we will show that if $\mathbf{b}'_i$ is not feasible, i.e., it violates the no-overbidding assumption, then $v_i(x_i(\mathbf{b}) + \ell) < \sum_{j=1}^{x_i(\mathbf{b})+\ell} b'_{ij}$. In this case, there must exist an index $t \leq x_i(\mathbf{b}) + \ell$, such that $v_i(t) < \sum_{j=1}^{t} b'_{ij}$. Note also that $t > r$, because $b'_{ij} = b_{ij}$ for $j \leq r$ and $\mathbf{b}$ is a feasible bidding vector. Assume that $t < x_i(\mathbf{b}) + \ell$ since, otherwise, we are done. We can decompose the sum of bids in our inequality as:

$$v_i(t) < \sum_{j=1}^{t} b'_{ij} = \sum_{j=1}^{r} b'_{ij} + \sum_{j=r+1}^{t} b'_{ij} = \sum_{j=1}^{r} b_{ij} + (t-r)(\beta_\ell(\mathbf{b}_{-i}) + \epsilon)$$

By rearranging the terms we obtain:

$$\begin{aligned}
(t-r)(\beta_\ell(\mathbf{b}_{-i}) + \epsilon) &> v_i(t) - \sum_{j=1}^{r} b_{ij} \\
&= v_i(t) - v_i(r) + v_i(r) - \sum_{j=1}^{r} b_{ij} \\
&\geq v_i(t) - v_i(r) = \sum_{j=r+1}^{t} m_i(j)
\end{aligned}$$

This means that there exists an index $s \in \{r+1, \dots, t\}$ such that $m_{is} < \beta_\ell(\mathbf{b}_{-i}) + \epsilon$. Then, by definition of $\mathbf{b}'_i$ and by the non-increasing marginal values of the submodular valuation function, we derive that $m_{ij} < b'_{ij}$, for $j = s+1, \dots, x_i(\mathbf{b}) + \ell$. Hence, since the no-overbidding assumption was violated at index t, it will continue to be violated if we include all the non-zero bids of $\mathbf{b}'_i$ up until the index $x_i(\mathbf{b}) + \ell$. Thus, $v_i(x_i(\mathbf{b}) + \ell) < \sum_{j=1}^{x_i(\mathbf{b})+\ell} b'_{ij}$, which concludes the proof of the claim. □

Case 2: Suppose $\mathbf{b}'_i$ is not feasible, i.e., it involves overbidding. Then we can still infer a lower bound on $\beta_\ell(\mathbf{b})$, by exploiting Claim 4.1, as follows:

$$
\begin{aligned}
v_i(x_i(\mathbf{b}) + \ell) < \sum_{j=1}^{x_i(\mathbf{b})+\ell} b'_{ij} &= \sum_{j=1}^{r} b_{ij} + (x_i(\mathbf{b}) + \ell - r) \cdot \Big(\beta_\ell(\mathbf{b}_{-i}) + \epsilon\Big) \\
&\leq \sum_{j=1}^{x_i(\mathbf{b})} b_{ij} + (x_i(\mathbf{b}) + \ell) \cdot \Big(\beta_\ell(\mathbf{b}_{-i}) + \epsilon\Big) \qquad (4.7) \\
&\leq v_i(x_i(\mathbf{b})) + (x_i(\mathbf{b}) + \ell) \cdot \Big(\beta_\ell(\mathbf{b}_{-i}) + \epsilon\Big) \qquad (4.8)
\end{aligned}
$$

where the last inequality holds because $\mathbf{b}$ is a feasible profile. By rearranging, we obtain:

$$
\beta_\ell(\mathbf{b}_{-i}) > \frac{1}{\ell + x_i(\mathbf{b})} \cdot \Big(v_i(x_i(\mathbf{b}) + \ell) - v_i(x_i(\mathbf{b}))\Big) - \epsilon.
$$

Observe that the above *strict* inequality holds for any arbitrarily small constant $\epsilon > 0$. Since also $\beta_\ell(\mathbf{b}_{-i}) \leq \beta_\ell(\mathbf{b})$, inequality (4.4) follows and the proof is concluded. □

4.1.4 *A Matching Lower Bound*

We now present a lower bound construction, establishing that our upper bound is tight.

Theorem 4.8. *The Price of Anarchy of the Uniform Price Auction for pure Nash equilibria and submodular bidders is at least*

$$
1 + \frac{(1 - \frac{1}{k})(1 + \mathcal{W}_0(-e^{-2}))}{\frac{1}{k-1} + 1 + (-\mathcal{W}_0(-e^{-2})(1 - \frac{1}{k}) - \frac{1}{k}) \ln(-\mathcal{W}_0(-e^{-2}) + \frac{1}{k})}
$$

and approaches $(2 + \mathcal{W}_0(-e^{-2}))/(1 + \mathcal{W}_0(-e^{-2})) \approx 2.188$ *as k grows.*

Proof. We construct an instance of the Uniform Price Auction with two bidders and $k \geq 8$ units. Let $x \in \{1, 2, \dots, k-2\}$ be a parameter that we will set later on. The valuation of bidder 1 assigns value only for the first unit and equals

$$
m_{11} = \frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i}
$$

For the remaining units, we have $m_{1j} = 0$, for any $j \geq 2$.

The valuation function of bidder 2 is given by the following marginal values:

$$m_{2j} = \begin{cases} 1, & 1 \leq j \leq k-1 \\ 0, & j = k \end{cases}$$

Hence, the optimal allocation is for bidder 1 to obtain 1 unit and for bidder 2 to obtain $k-1$ units.

Now, consider a bidding profile $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2)$ defined as follows:

$$b_{1j} = \begin{cases} 1 - \frac{x}{k-1}, & j = 1 \\ 1 - \frac{x}{k-j+1}, & j = 2, \ldots, k-x \\ 0, & \text{otherwise} \end{cases}$$

and

$$b_{2j} = \begin{Bmatrix} \epsilon, & j = 1, \ldots, x \\ 0, & j > x \end{Bmatrix}.$$

Here $\epsilon > 0$ is an arbitrarily small positive quantity.

We will see that this construction yields better lower bounds than the previously known bound of $2 - \frac{1}{k}$, when $k \geq 11$. For example, for $k = 11$ and $x = 2$ we obtain the following instance:

$$\mathbf{m}_1 = (5.942, 0, 0, \ldots, 0, 0, 0), \quad \mathbf{b}_1 = (\frac{8}{10}, \frac{8}{10}, \frac{7}{9}, \frac{6}{8}, \frac{5}{7}, \ldots, \frac{1}{3}, 0, 0)$$
$$\mathbf{m}_2 = (1, 1, \ldots, 1, 1, 0), \quad \mathbf{b}_2 = (\epsilon, \epsilon, 0, 0, \ldots, 0, 0)$$

It can be shown that this instance already yields a lower bound on the Price of Anarchy of 2.007. Coming back now to the analysis for general k and x, we will first ensure that both bidding vectors $\mathbf{b}_1$,$\mathbf{b}_2$ adhere to no-overbidding. For the vector $\mathbf{b}_1$, it suffices to note that

$$\sum_{j=1}^{k-x} b_{1j} = \frac{k-1-x}{k-1} + \sum_{j=2}^{k-x} \frac{k-j+1-x}{k-j+1} = \frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i} = m_{11}$$

where the last equality holds by changing indices and setting $i = k - j + 1 - x$. Therefore, we have that $\sum_{j=1}^{k-x} b_{1j} = v_1(k-x)$. And this directly implies that for any $\ell < k - x$, we have $\sum_{j=1}^{\ell} b_{1j} < v_1(\ell)$. It is also straightforward that for $\ell > k - x$, the no-overbidding assumption cannot be violated. Similarly, for the vector $\mathbf{b}_2$, it is easy to check that it complies to no-overbidding.

Under $\mathbf{b}$, bidder 1 obtains $k - x$ units and bidder 2 obtains x units. Notice that in this profile the uniform price is 0, as there is no contest for any unit; bidder 1 bids for exactly $k - x$ units, while bidder 2 bids for x units. All other bids are 0.

We now argue that $\mathbf{b}$ is a pure Nash equilibrium, under the assumption that whenever there is a tie in a deviation from $\mathbf{b}$, bidder 1 always gets the unit in question. Bidder 1 clearly has no incentive to deviate. She is interested only in the first unit, and there is no incentive to win more units for her. Note also that she cannot lose the first unit (even if she bids a zero vector) due to the tie breaking rule.

Let us examine the case of bidder 2. Since bidder 2 is not interested in the last unit, we can consider only deviation vectors $\mathbf{b}'_2$ with $b'_{2k} = 0$. Note that under $\mathbf{b}$, $u_2(\mathbf{b}) = x$. Hence, bidder 2 does not have an incentive to try to obtain less than x units, since the price will then still remain 0, and she will only have lower utility. It therefore suffices to consider what happens when she tries to obtain ℓ additional units, where $\ell = 1, \ldots, k - x - 1$. To do so, bidder 2 must outbid some of the winning bids of $\mathbf{b}_1$. In particular, to obtain ℓ additional units at the minimum possible price, she must outbid the bid b_{1t} of bidder 1, where t is the index $t = k - x - (\ell - 1)$. If she issues a bid $\mathbf{b}'_2$, where the first $x + \ell$ coordinates outbid b_{1t} and the remaining bids are 0, then she will obtain exactly $x + \ell$ units, and the new price (i.e., the new highest losing bid) will be precisely b_{1t}. However, any such attempt will grant bidder 2 utility equal to $u_2(\mathbf{b})$, since

$$\begin{aligned} u_2(\mathbf{b}_1, \mathbf{b}'_2) &= v(x + \ell) - (x + \ell) \cdot b_{1t} \\ &= x + \ell - (x + \ell) \cdot (1 - \frac{x}{x + \ell}) = x = u_2(\mathbf{b}). \end{aligned}$$

We conclude that the profile $\mathbf{b}$ is a pure Nash Equilibrium. The ratio of the optimal Social Welfare to the one in $\mathbf{b}$ is at least:

$$\begin{aligned} &\frac{SW(\mathbf{x}^*)}{SW(\mathbf{b})} = \frac{v_1(1) + v_2(k-1)}{v_1(k-x) + v_2(x)} \\ &= 1 + \frac{k - 1 - x}{\frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i} + x} = 1 + \frac{k - 1 - x}{\frac{k-1-x}{k-1} + k - 1 - x \sum_{i=x+1}^{k-1} \frac{1}{i}} \\ &\geq 1 + \frac{k - 1 - x}{\frac{k-1-x}{k-1} + k - 1 - x \int_{x+1}^{k} \frac{1}{y} dy} \geq 1 + \frac{k - 1 - x}{\frac{k-1-x}{k-1} + k - 1 - x \ln \frac{k}{x+1}} \end{aligned} \tag{4.9}$$

At this point we set $x = \lfloor -\mathcal{W}_0(-e^{-2})\dot{(}k-1) \rfloor$, where $\mathcal{W}_0$ is the first branch of the Lambert W function. To continue from (4.9), we will need to ensure that $-\mathcal{W}_0(-e^{-2})(k-1) - 1 > 0$, which holds for $k \geq 8$. We now have:

$$(4.9) = 1 + \frac{k - 1 - \lfloor -\mathcal{W}_0(-e^{-2})(k-1) \rfloor}{\frac{k-1-\lfloor -\mathcal{W}_0(-e^{-2})(k-1) \rfloor}{k-1} + k - 1 - \lfloor -\mathcal{W}_0(-e^{-2})(k-1) \rfloor \ln \frac{k}{\lfloor -\mathcal{W}_0(-e^{-2})(k-1) \rfloor + 1}}$$

$$\geq 1 + \frac{k - 1 - (-\mathcal{W}_0(-e^{-2})(k-1))}{\frac{k-(-\mathcal{W}_0(-e^{-2})(k-1))}{k-1} + k - 1 - (-\mathcal{W}_0(-e^{-2})(k-1) - 1)\ln\frac{k}{-\mathcal{W}_0(-e^{-2})(k-1)+1}}$$
$$\geq 1 + \frac{(1-\frac{1}{k})(1+\mathcal{W}_0(-e^{-2}))}{\frac{1}{k-1} + 1 + (-\mathcal{W}_0(-e^{-2})(1-\frac{1}{k}) - \frac{1}{k})\ln(-\mathcal{W}_0(-e^{-2}) + \frac{1}{k})} = f(k). \tag{4.10}$$

So far, we have shown that

$$\frac{SW(\mathbf{x}^*)}{SW(\mathbf{b})} \geq f(k).$$

The theorem follows by observing that as k goes to ∞, the function $f(\cdot)$ becomes:

$$\lim_{k\to\infty} f(k) = 1 + \frac{1+\mathcal{W}_0(-e^{-2})}{1-\mathcal{W}_0(-e^{-2})\cdot\ln\left(-\mathcal{W}_0(-e^{-2})\right)} = \frac{2+\mathcal{W}_0(-e^{-2})}{1+\mathcal{W}_0(-e^{-2})},$$

where the last equality is derived by using the property that $\ln(-\mathcal{W}_0(y)) = -\mathcal{W}_0(y) + \ln(-y)$ for $y \in [-e^{-1}, 0)$, see Corless et al. [1996]. □

4.2 MIXED AND BAYES-NASH EQUILIBRIA

In this section we present upper bounds on the Price of Anarchy of Bayes-Nash Equilibria for the uniform price auction. Due to the hierarchy of equilibria classes, these upper bounds also hold for mixed equilibria.

4.2.1 *Submodular Valuation Functions*

We will perform another derivation for the upper bound of the Bayesian Price of Anarchy with submodular bidders of the Uniform Price auction. Our proof is inspired by the derivation of Christodoulou et al. [2016] for the discriminatory price auction. This derivation matches the already established upper bound by de Keijzer et al. [2013]. We believe that this derivation is useful even though it does not improve the current upper bound since it gives an insight on how a potential improvement could be achieved.

We begin by presenting a parameterized version of Lemma 3.3 from Christodoulou et al. [2016].

Lemma 4.9. *Let F be the cumulative distribution function (CDF) of a non-negative random variable p and $v > 0$ be a fixed constant. Define $a^* = \arg\max_a\{F(a)(v-a)\}$. For any $\mu > 0$, it is true that*

$$F(a^*)(v-a^*) + \mu\,\mathbb{E}[p] \geq \mu(1 - \frac{1}{e^{\frac{1}{\mu}}})v.$$

Proof. Set $A = F(a^*)(v - a^*)$, for $a^* = \arg\max_a \{F(a)(v-a)\}$. We will use the following fact from probability theory.

Fact 4.1. *For any non-negative random variable x with a CDF F it is true that*

$$\mathbb{E}[x] = \int_0^\infty (1 - F(y))dy.$$

Thus,

$$\begin{aligned}
F(a^*)(v-a^*) + \mu\,\mathbb{E}[p] &= A + \mu \int_0^\infty (1 - F(x))dx \\
&\geq A + \mu \int_0^{v-A} (1 - F(x))dx \\
&= A + \mu(v - A) - \mu \int_0^{v-A} F(x)dx \\
&\geq \mu v + (1-\mu)A - \mu \int_0^{v-A} \frac{A}{v-x}dx \\
&= \mu v + (1-\mu)A + \mu A \ln(\frac{A}{v}) \\
&= v\left(\mu + (1-\mu)\frac{A}{v} + \mu\frac{A}{v}\ln(\frac{A}{v})\right).
\end{aligned}$$

To continue, we minimize the function $g(y) = (1-\mu)y + \mu y \ln(y)$ over $(0,1)$, since A/v belongs to this interval.

Fact 4.2. *For a positive constant μ, the minimum of the function $g(y) = (1-\mu)y + \mu y \ln(y)$ over $(0,1)$, is achieved at $y = \frac{1}{e^{\frac{1}{\mu}}}$.*

By substituting, we obtain:

$$F(a^*)(v-a^*) + \mu\,\mathbb{E}[p] \geq \mu\left(1 - \frac{1}{e^{\frac{1}{\mu}}}\right)v,$$

and the lemma follows. □

At this point, let us introduce some auxiliary notation.

Let $\mathbf{b}$ be a randomized bidding profile drawn from distribution $\mathcal{P}$. For this distribution $\mathcal{P}$, $\beta_j(\mathbf{b})$ is also a random variable that depends on $\mathbf{b} \sim \mathcal{P}$. We define the following:

$$\begin{aligned}
F_{ij}(y) &= \Pr\left[\beta_j(\mathbf{b}_{-i}) \leq y\right], \quad \text{for } 1 \leq j \leq k \\
G_{ij}(y) &= \Pr[\beta_j(\mathbf{b}_{-i}) < y \leq \beta_{j+1}(\mathbf{b}_{-i})] = F_{ij}(y) - F_{i(j+1)}(y), \\
&\qquad \text{for } 1 \leq j \leq k-1.
\end{aligned}$$

We separately define $G_{ik}(y) = \Pr[y \geq \beta_k]$. For every $i \in \mathcal{N}$ and $j = 1, \ldots, k$, F_{ij} is the *cumulative distribution function* (CDF) of $\beta_j(\mathbf{b}_{-i})$. Additionally, for the function G_{ij} it holds that

$$F_{ij}(y) = \sum_{\ell=j}^{k} G_{i\ell}(y),$$

$$\sum_{j=1}^{\ell} F_{ij}(y) = \sum_{j=1}^{\ell} jG_{ij}(y) + \sum_{j=\ell+1}^{k} \ell G_{ij}(y) \tag{4.11}$$

$$\sum_{j=\ell}^{k} G_{ij}(y) \cdot \mathbb{E}[\beta_\ell | \beta_\ell < y \leq \beta_{\ell+1}] = F_{i\ell}(y) \cdot \mathbb{E}[\beta_\ell | \beta_\ell < y]. \tag{4.12}$$

Let $F_i^{\text{avg}}(y) = \frac{1}{x_i^{\mathbf{v}}} \sum_{j=1}^{x_i^{\mathbf{v}}} F_{ij}(y)$ be the cumulative distribution function of a random variable called β_i^{avg}. Note that F_i^{avg} is a CDF defined on R^+ since $F_i^{\text{avg}}(0) = 0$, $\lim_{y\to\infty} F_i^{\text{avg}}(y) = 1$ and $F_i^{\text{avg}}(y)$ is a non-decreasing function,being the average of non-decreasing CDFs.

Additionally,

$$\mathbb{E}[\beta_i^{\text{avg}}] = \int_0^\infty (1 - F_i^{\text{avg}}(x))dx = \int_0^\infty (1 - \frac{1}{x_i^{\mathbf{v}}} \sum_{j=1}^{x_i^{\mathbf{v}}} F_{ij}(x))dx$$
$$= \frac{1}{x_i^{\mathbf{v}}} \int_0^\infty (1 - F_{ij}(x))dx = \frac{1}{x_i^{\mathbf{v}}} \mathbb{E}_{\mathbf{b}_{-i}\sim\mathcal{P}_{-i}} \left[\sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}) \right].$$

We can now present an alternative proof of the lemma 3.8 inspired by the non-smooth approach of Christodoulou et al. [2016].

Lemma 4.10. *For any submodular valuation profile* $\mathbf{v}$ *and any randomized bidding profile* B*, there exists a pure bidding strategy* $\mathbf{b}'_i$ *for each player* $i \in \mathcal{N}$ *such that, for every* $\mu > 0$*, it holds that*

$$\mathbb{E}_{\mathbf{b}_{-i}\sim\mathcal{P}_{-i}}[u_i^{v_i}(\mathbf{b}'_i, \mathbf{b}_{-i})] \geq \mu \left(1 - \frac{1}{e^{\frac{1}{\mu}}}\right) \cdot v_i(x_i^{\mathbf{v}}) - \mu \cdot \mathbb{E}_{\mathbf{b}_{-i}\sim\mathcal{P}_{-i}} \left[\sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}) \right].$$

Proof. Let $\mathbf{b}$ be a randomized bidding profile drawn from distribution $\mathcal{P}$. Fix a bidder $i \in \mathcal{N}$ and a valuation profile profile $\mathbf{v}$. Let

$$a^* = \arg\max_a \{F_i^{\text{avg}}(a) \left(\frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} - \alpha \right)\}.$$

Assume bidder i performs a unilateral deviation bidding the following bid vector:

$$\mathbf{b}'_i = \left(\underbrace{a^*, a^*, \ldots, a^*}_{x_i^{\mathbf{v}} \text{ bids}}, 0, \ldots, 0 \right)$$

Note that $\mathbf{b}'_i$ adheres to the no-overbidding assumption since for every $\ell = 1, \ldots, x_i^{\mathbf{v}}$ it holds that

$$\ell \cdot a^* \leq \ell \cdot \frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} \leq v(l),$$

where the first inequality holds due to the fact that $a^* \leq \frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}}$ and the second inequality by proposition 2.2.

Let $\mathcal{E}_j(a^*)$ be the event that $\beta_j(\mathbf{b}_{-i}) \leq a^* \leq \beta_{j+1}$, for $j = 1, \ldots, k-1$. Additionally, let $\mathcal{E}_k(a^*)$ be the event that $a^* \geq \beta_k$.

With the law of total expectation as a starting point, we have:

$$\begin{aligned}
\mathbb{E}[u_i^{v_i}(\mathbf{b}'_i, \mathbf{b}_{-i})] &\geq \sum_{j=1}^{x_i^{\mathbf{v}}-1} G_{ij}(a^*)(v(j) - ja^*) \\
&\quad + \sum_{j=x_i^{\mathbf{v}}}^{k} G_{ij}(a^*)(v(x_i^{\mathbf{v}}) - x_i^{\mathbf{v}}\, \mathbb{E}[\beta_{x_i^{\mathbf{v}}}(\mathbf{b}_{-i}) | \mathcal{E}_j(a^*)] \\
&\geq \sum_{j=1}^{x_i^{\mathbf{v}}-1} G_{ij}(a^*) \left(\frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} - a^* \right) j \\
&\quad + x_i^{\mathbf{v}} \sum_{j=x_i^{\mathbf{v}}}^{k} G_{ij}(a^*) \frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} - x_i^{\mathbf{v}} \sum_{j=x_i^{\mathbf{v}}}^{k} G_{ij}(a^*)\, \mathbb{E}[\beta_{x_i^{\mathbf{v}}}(\mathbf{b}_{-i}) | \mathcal{E}_j(a^*)]
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&= \sum_{j=1}^{x_i^{\mathbf{v}}-1} G_{ij}(a^*) \left(\frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} - a^* \right) j + x_i^{\mathbf{v}} \sum_{j=x_i^{\mathbf{v}}}^{k} G_{ij}(a^*) \frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} \\
&\quad - x_i^{\mathbf{v}} F_{i(x_i^{\mathbf{v}})}(a^*)\, \mathbb{E}[\beta_{x_i^{\mathbf{v}}}(\mathbf{b}_{-i}) | a^* > \beta_{x_i^{\mathbf{v}}}(\mathbf{b}_{-i})]
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&= \sum_{j=1}^{x_i^{\mathbf{v}}} F_{ij}(a^*) \left(\frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} - a^* \right) j + x_i^{\mathbf{v}} F_{i(x_i^{\mathbf{v}})}(a^*) a^* \\
&\quad - x_i^{\mathbf{v}} F_{i(x_i^{\mathbf{v}})}(a^*)\, \mathbb{E}[\beta_{x_i^{\mathbf{v}}}(\mathbf{b}_{-i}) | a^* > \beta_{x_i^{\mathbf{v}}}(\mathbf{b}_{-i})]
\end{aligned} \tag{4.15}$$

$$= \sum_{j=1}^{x_i^{\mathbf{v}}} F_{ij}(a^*) \left(\frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} - a^* \right) j + x_i^{\mathbf{v}} \int_0^{a^*} F_{i(x_i^{\mathbf{v}})}(y) dy \tag{4.16}$$

$$= x_i^{\mathbf{v}} \left(\left(\frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} - a^* \right) F_i^{\text{avg}}(a^*) + \mu \, \mathbb{E}[\beta_i^{\text{avg}}] + \int_0^{a^*} F_{i(x_i^{\mathbf{v}})}(y) dy \right) \tag{4.17}$$

$$- \mu x_i^{\mathbf{v}} \, \mathbb{E}[\beta_i^{\text{avg}}]$$

$$\geq x_i^{\mathbf{v}} \left(\mu \frac{v(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}} (1 - \frac{1}{e^{\frac{1}{\mu}}}) \right) - \mu x_i^{\mathbf{v}} \, \mathbb{E}[\beta_i^{\text{avg}}] \tag{4.18}$$

$$= \mu \left(1 - \frac{1}{e^{\frac{1}{\mu}}} \right) \cdot v_i(x_i^{\mathbf{v}}) - \mu \cdot \mathbb{E}_{\mathbf{b}_{-i} \sim \mathcal{P}_{-i}} \left[\sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}) \right].$$

Inequality (4.13) follows by the first statement of proposition 2.2 for submodular valuation functions. Equalities 4.14 and (4.15) are due to equations (4.12) and (4.11), whereas (4.16) follows from the fact that $\mathbb{E}[p | p < c] = c - \frac{1}{CDF(c)} \cdot \int_0^c CDF(y) dy$, for every non negative random variable p and every $c > 0$. Finally, inequality (4.18) follows from lemma 4.9. □

We can now show that this derivation matches the upper bound presented byde Keijzer et al. [2013].

Theorem 4.11. *The Bayesian Price of Anarchy of the Uniform Price Auction (under the standard or uniform bidding format) is at most* $-\mathcal{W}_{-1}(-e^{-2}) \approx 3.146$ *for submodular valuation functions, where* $\mathcal{W}_{-1}$ *is the lower branch of the Lambert function.*

Proof. With lemma 4.10 we have shown that equation (3.4) from theorem 3.7 can be satisfied for

$$(\lambda, \mu) = \left(\mu \left(1 - \frac{1}{e^{\frac{1}{\mu}}}, \mu \right) \right).$$

Therefore the Price of Anarchy of the Uniform Price Auction is at most

$$\frac{\mu + 1}{\mu (1 - \frac{1}{e^{\frac{1}{\mu}}})}.$$

The expression is maximized at $\mu = \frac{-1}{2 + \mathcal{W}_{-1}(-e^{-2})} \approx 0.87$ and thus, the theorem follows. □

Note that this result holds for both the standard and the uniform bidding interface.

Remark 4.1. *Notice that in lemma 4.10 we omitted the non-negative term* $xi^{\mathbf{v}} \int_0^{a^*} F_{i(x_i^{\mathbf{v}})}(y) dy$*. Possibly, finding an improved lower bound for this term in terms of valuation could lead to an improved upper bound for the Bayesian Price of Anarchy. Alternatively, understanding when this term is close to* 0 *could lead to an improved lower bound instead.*

Weak Smoothness In chapter 3 we briefly discussed the connection of theorem 3.7 with the Smoothness Framework of Syrgkanis and Tardos [2013]. In contrast to the discriminatory price auction, the uniform is not a (λ, μ)-smooth mechanism. Rather, it can be shown that it adheres to the definition of Syrgkanis and Tardos [2013] for a weakly $(\lambda, \mu_1\mu_2)$-smooth mechanism.

Definition 4.1 (Syrgkanis and Tardos [2013]). *A mechanism $\mathcal{M}$ is weakly $(\lambda, \mu_1\mu_2)$-smooth for $\lambda > 0$ and $\mu_1, \mu_2 \geq 0$ if for any valuation profile $\mathbf{v}$ and for any bidding profile $\mathbf{b}$ there exists a randomized bidding profile $B_i'(\mathbf{v}, \mathbf{b}_i)$ for each bidder i, such that*

$$\sum_{i \in \mathcal{N}} \mathbb{E}_{\mathbf{b'}_i \sim B_i'} \left[u_i^{\mathbf{v}_i}(\mathbf{b'}_i, \mathbf{b}_{-i}) \right] \geq \lambda \cdot SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu_1 \cdot \sum_{i \in \mathcal{N}} P_i(\mathbf{b}) \\ - \mu_2 \cdot \sum_{i \in \mathcal{N}} W_i(\mathbf{b}_i, x_i(\mathbf{b})),$$

where $W_i(\mathbf{b}_i, x_i(\mathbf{b})) = \max_{\mathbf{b}_{-i} = x_i(\mathbf{b})} P_i(\mathbf{b})$ is i's willingness to pay.

This definition of smoothness is appropriate for auction mechanisms where a no-overbidding assumption is necessary for the analysis. The uniform price auction is one of those mechanisms and it can be shown using lemma 3.8 or 4.10 that it is weakly smooth for $(\lambda, \mu_1, \mu_2) = ((\mu(1 - \frac{1}{e^{\frac{1}{\mu}}}, 0, \mu))$.

4.2.2 *Subadditive Valuation Functions*

To derive welfare guarantees under the setting of incomplete information for subadditive bidders and both bidding interfaces, we will once again utilize theorem 3.7. By combining 4.10, equation (3.3) and theorem 3.7 we conclude that the Bayesian Price of Anarchy is at most $2\mathcal{W}_{-1}(-e^{-2}) \approx 6.292$.

For the standard bidding interface solely, similarly to the discriminatory price auction, we have the following lemma.

Lemma 4.12 (de Keijzer et al. [2013]). *Let V be the class of subadditive valuation functions. Then, Theorem 3 holds true with $(\lambda, \mu) = (\frac{1}{2}, 1)$ for the Uniform Price Auction.*

Thus, by theorem 3.7, we conclude that the Bayesian Price of Anarchy is at most 4 under the standard bidding interface.

5 CONCLUSIONS AND OPEN PROBLEMS

There are still several intriguing open questions for future research in multi-unit auctions.

An open question for both auction formats is whether existence of mixed Nash and Bayes-Nash equilibria can be guaranteed. Recall the assumption 2.1 from chapter 2. In order to study the Price of Anarchy of mixed Nash and Bayes-Nash equilibria we have assumed that the continuous bidding space is bounded and finite (sufficiently discretized). The question is whether we can guarantee existence of these solution concepts *ex ante*, without the need to invoke this assumption.

A major open problem is to tighten the known gaps in the Price of Anarchy for the set of Bayes-Nash equilibria. Recall that for Bayesian Price of Anarchy the best known lower and upper bounds for submodular bidders are 1.109 and 1.54, due to de Keijzer et al. [2013] and Christodoulou et al. [2016] for the discriminatory price auction and 2.188 and 3.146[1] due to Birmpas et al. [2017]; de Keijzer et al. [2013]. Note that these bounds hold for both the standard and uniform bidding interfaces. For subadditive bidders, as shown by de Keijzer et al. [2013], there is still a gap for the discriminatory price auction under uniform bidding between 2 and 3.16. For the uniform price auction the problem is open for both bidding interfaces for which we only know the bounds of 2 and 6.292 for the uniform bidding interface and 2.188 and 4 for the standard bidding interface.

For the class of Pure Equilibria, it has been shown that the discriminatory price auction is efficient for all valuations and both bidding interfaces in de Keijzer et al. [2013], whereas the Price of Anarchy for pure Nash equilibria is now known to be almost 2.188 (main contribution of this thesis, Birmpas et al. [2017]) for submodular bidders for the standard bidding interface. For the uniform bidding interface, the currently known bounds are 2 and 3.146 due to de Keijzer et al. [2013].

We have tried deriving lower bounds for mixed Nash equilibria of the uniform price auction that are worse than 2.188 without success. Our efforts were based on extending the construction of theorem 4.8 having bidder 1 bid in a true probabilistic manner. Consider the following bidding profile:

$$\begin{aligned} \mathbf{m}_1 &= (1.833, 0, 0, 0), & \mathbf{b}_1 &= (y^{\frac{1}{2}}, y^{\frac{1}{2}}, y, 0) \\ \mathbf{m}_2 &= (1, 1, 1, 0), & \mathbf{b}_2 &= (\epsilon, 0, 0, 0) \end{aligned}$$

1 The lower bound for the Bayes-Nash equilibria of the uniform Price is one of our contributions. It holds for the Bayes-Nash equilibria since a Pure Nash equilibrium can also be regarded as a Bayes-Nash equilibrium.

In this example y is drawn uniformly at random from $[0,1]$. It can be shown that this is a mixed Nash equilibrium with a Price of Anarchy of 1.705. In fact, this instance is equivalent to the following pure Nash equilibrium:

$$\mathbf{m}_1 = (1.833, 0, 0, 0), \quad \mathbf{b}_1 = (\frac{2}{3}, \frac{2}{3}, \frac{1}{2}, 0)$$
$$\mathbf{m}_2 = (1, 1, 1, 0), \quad \mathbf{b}_2 = (\epsilon, 0, 0, 0)$$

If one could hope closing the gap between 2.188 and 3.146 by deriving a worse lower bound in terms of price of anarchy, randomization should be exploited. It is not obvious to us how this can be done using the construction above. Alternatively, recall remark 4.1. Can we exploit this loss for a lower upper bound?

Regarding subadditive bidders, there are no guarantees for the existence of Pure equilibria (recall that the claim 4.1 only holds for submodular valuations). Finally, it still remains elusive to produce lower bounds tailored for subadditive functions, and the best known upper bound is 4, by de Keijzer et al. [2013].

Finally, from a mechanism design perspective, an interesting direction of research would be to design new protocols that admit even more efficient equilibria. A desirable feature of these potential new mechanisms is the simplicity of the proposed pricing scheme.

6

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